Two examples of reaction-diffusion front propagation in heterogeneous media

Soutenance de thèse

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Soutenue par le projet ERC ReaDi
1. Fisher-KPP propagation driven by a line of fast diffusion: non-local exchanges
   - Presentation of the model
   - Results
     - Robustness of the BRR-model
     - Specific Properties of the model
     - Uniform dynamics under a singular limit
   - Conclusion

2. Bistable entire solutions
   - Introduction
   - Previous and current results
     - Entire solution in cylinder-like domains
     - A one dimensional case study
   - Conclusion
Nonlocal model

Model under study

\[
\begin{aligned}
&\partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t, x, y) dy \quad x \in \mathbb{R}, \ t > 0 \\
&\partial_t v - d\Delta v = f(v) + \mu(y) u(t, x) - \nu(y) v(t, x, y) \quad (x, y) \in \mathbb{R}^2, \ t > 0
\end{aligned}
\]  

(1)

Hypotheses:

- The function \( f \) is of KPP-type: \( f(0) = f(1) = 0, \ f \) nonnegative, concave on \([0, 1]\). Introduced by A. Kolmogorov, I. Petrovsky, and N. Piskounov (1937). In: Bull. Univ. Etat Moscou for the equation \( \partial_t u - \partial_{xx} u = u(1 - u) \).

- \( \nu, \mu \geq 0 \), continuous, even and compactly supported. \( \overline{\mu} = \int \mu, \overline{\nu} = \int \nu \).

The functions \( \nu \) and \( \mu \) model exchanges of densities between the road and the field → exchange functions.
Initial question

Enhancement of biological invasion by heterogeneities: effect of a line of fast diffusion with nonlocal exchanges.

\[ \partial_t u - D \partial_{xx} u = \text{exchange terms} \]

The Field

Road of fast diffusion: \( \partial_t u - D \partial_{xx} u = \text{exchange terms} \)

The Field

Exchanges area (support of \( \mu \) or \( \nu \))

nonlocal equation

KPP Reaction-Diffusion

\[ \partial_t v - d \Delta v = f(v) \]
Biological motivation

Influence of transportation network on biological invasion

Seismic lines in Alberta forest. Copyright (c) Province of British Columbia. All rights reserved. Reproduced with permission of the Province of British Columbia.
GPS observation: wolves move and concentrate along these lines

Initial model


\[
\partial_t u - D \partial_{xx} u = \bar{v} v - \bar{\mu} u
\]

\[
d \partial_y v = \mu u - \nu v
\]

\[
\partial_t v - d \Delta v = f(v)
\]

KPP reaction-diffusion

Our model deals with nonlocal exchange terms.
Initial model

Berestycki-Roquejoffre-Rossi

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u - D \partial_{xx} u = \bar{v} v(t, x, 0) - \bar{\mu} u \quad x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d \Delta v = f(v) \quad (x, y) \in \mathbb{R} \times \mathbb{R}^*, \ t > 0 \\
\end{array} \right.
\end{aligned}
\]  \tag{2}

\[
\left\{ \begin{array}{l}
v(t, x, 0^+) = v(t, x, 0^-), \quad x \in \mathbb{R}, \ t > 0 \\
-d \left\{ \partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-) \right\} = \bar{\mu} u(t, x) - \bar{v} v(t, x, 0) \quad x \in \mathbb{R}, \ t > 0.
\end{array} \right.
\]  \tag{3}

- Faster diffusion on the road: $D > d$.
- Exchange coefficients at the boundary: $\bar{\mu}, \bar{v}$.
- Reaction term $f$ of KPP type.

Initial question

Does the road enhance the spreading?
The homogeneous case

Reaction-diffusion with KPP nonlinearity:

$$\partial_t u = d\Delta u + f(u).$$

(4)

Theorem - definition


Let $u(t, x)$ be solution of (4) with $0 \leq u_0 \leq 1$, compactly supported. Then there exists $c^*$ such that

- $\forall c > c^*, \lim_{t \to \infty} \sup_{|x| > ct} u(t, x) = 0$
- $\forall c < c^*, \lim_{t \to \infty} \inf_{|x| < ct} u(t, x) = 1$

with $c^* = 2\sqrt{df'(0)}$.

In homogeneous media, propagation in every direction at speed $c_{KPP} := 2\sqrt{df'(0)}$.

Expansion of the muskrat in Europe.

J. G. SKELLAM (1951). In: *Biometrika*
Results of Berestycki-Roquejoffre-Rossi

Theorem


There exists $c^* = c^* (\mu, \nu, d, D) > 0$ such that:

- for all $c > c^*$, \( \lim_{t \to \infty} \sup_{|x| \geq ct} (u(t, x), v(t, x, y)) = (0, 0) \); 
- for all $c < c^*$, \( \lim_{t \to \infty} \inf_{|x| \leq ct} (u(t, x), v(t, x, y)) = (\nu/\mu, 1) \).

Moreover:

- if $D \leq 2d$, then $c^*(\mu, \nu, d, D) = c_{KPP} := 2 \sqrt{df'(0)}$;
- if $D > 2d$, then $c^*(\mu, \nu, d, D) > c_{KPP}$ and $\lim_{D \to \infty} c^*(\mu, \nu, d, D)/\sqrt{D}$ exists and is positive.

- Enhancement of the spreading in the direction of the road.
- Threshold $D = 2d$. 

Enhancement of the spreading in the direction of the road.
Questions

1. Can we retrieve the same kind of results for the nonlocal model?
2. How do nonlocal exchanges modify the spreading speed?
3. How can we retrieve the initial model from the nonlocal one?
Fisher-KPP propagation driven by a line of fast diffusion: non-local exchanges

- Presentation of the model
- Results
  - Robustness of the BRR-model
  - Specific Properties of the model
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Bistable entire solutions

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Stationary solutions

Proposition

(1) admits a unique nonnegative bounded stationary solution \((U_s, V_s(y)) \neq (0, 0)\). This solution is \(x\)–invariant.

\[
\begin{align*}
U_s & = \frac{1}{\mu} \int \nu(y) V_s(y) dy \\
-dV_s''(y) & = f(V_s(y)) + U_s \mu(y) - V_s(y) \nu(y)
\end{align*}
\]

\(V_s(\pm \infty) = 1\).

Reminder: in the initial BRR-case, \((U_s, V_s) = \left( \frac{\nu}{\mu}, 1 \right)\).
Robustness of the BRR-result

**Theorem**

There exists $c^* = c^*(\mu, \nu, d, D, f'(0)) > 0$ such that:

- For all $c > c^*$, \( \lim_{t \to \infty} \sup_{|x| \geq ct} (u(t, x), v(t, x, y)) = (0, 0) \);
- For all $c < c^*$, \( \lim_{t \to \infty} \inf_{|x| \leq ct} (u(t, x), v(t, x, y)) = (U_s, V_s) \).

Moreover, $c^*$ satisfies:

- If $D \leq 2d$, $c^* = c_{KPP} := 2 \sqrt{df'(0)}$;
- If $D > 2d$, $c^* > c_{KPP}$.

**Remark**

The threshold is still $D = 2d$. 
Main tool: construction of planar waves

They serve as supersolutions \( f(v) \leq f'(0)v \).

**Linearised system**

\[
\begin{cases}
\partial_t u - D \partial_{xx} u = -\mu u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, \\
\partial_t v - d\Delta v = f'(0)v + \mu(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2,
\end{cases}
\] (5)

**Exponential solutions of the form**

\[
\begin{pmatrix}
u(t, x) \\
v(t, x, y)
\end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\
\phi(y)
\end{pmatrix},
\] (6)

With nonnegative \( \lambda, c, \phi \in H^1(\mathbb{R}) \).

- In the BRR model, they were given by an algebraic computation.
- Here we are led to a nonlinear eigenvalue problem.
Equivalent system in $\lambda, \phi, c$

\[
\begin{cases}
-D\lambda^2 + \lambda c + \bar{\mu} = \int \nu(y)\phi(y)dy \\
-d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y).
\end{cases}
\]

- First equation gives a map $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \bar{\mu}$.
- Second equation: at most one solution $\phi = \phi(y; \lambda, c)$. Then set $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$.

**Goal**

Find $\lambda, c$ such that the graphs of $\lambda \mapsto \Psi_1(\lambda)$ and $\lambda \mapsto \Psi_2(\lambda)$ intersect.

**Proposition**

If $c > c_{KPP}$, then:

1. $\lambda \mapsto \Psi_2(\lambda)$ defined on $]\lambda_2^-, \lambda_2^+$ is positive, smooth, convex and symmetric with respect to the line $\{\lambda = \frac{c}{2d}\}$.
2. $\Psi_2(\lambda) \xrightarrow[\lambda \to \lambda_2^+] \bar{\mu}$, $\frac{d\Psi_2}{d\lambda}(\lambda) \xrightarrow[\lambda \to \lambda_2^-] -\infty$. 
The two curves $\Psi_1$ and $\Psi_2$
Behaviour as $c$ increases

\[ \rho \mu \]

\[ \psi_1 \quad \psi_2 \]
Intersection for $c = c^*$
Slower spreading

\[ \lambda \]

\[ \Psi_1 \]

\[ \Psi_2 \]
Faster spreading

\[ \psi_1 \quad \psi_2 \]

\[ \bar{\mu} \quad \lambda \]
Natural question: influence of nonlocal exchanges on the spreading speed

For fixed parameters $d, D, f'(0), \mu, \nu$ we consider the set of admissible exchanges

$$\Lambda_{\mu} = \{ \mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \mu, \mu \text{ even} \}.$$

Reminder: for $\mu \in \Lambda_{\mu}$ and $\nu \in \Lambda_{\nu}$, there exists a spreading speed $c^*(\mu, \nu)$. Let $c_0^*$ be the spreading speed for the initial BRR model ($c_0^* = c^*(\mu_0, \nu_0)$).

Questions

- Can we compare $c^*(\mu, \nu)$ with $c_0^*$?
- $\inf\{c^*(\mu, \nu), \mu \in \Lambda_{\mu}, \nu \in \Lambda_{\nu}\}$?
- $\sup\{c^*(\mu, \nu), \mu \in \Lambda_{\mu}, \nu \in \Lambda_{\nu}\} = c_0^*$?
Long range exchange terms: a new threshold

For fixed parameters $d, D, f'(0), \mu, \nu$ we can get the infimum with

$$\nu_R(y) = \frac{1}{R} \nu \left( \frac{y}{R} \right), \text{ or } \mu_R(y) = \frac{1}{R} \mu \left( \frac{y}{R} \right), \quad R \to +\infty$$

**Theorem**

Let us consider the nonlocal system (1) with fixed exchange masses $\mu$ and $\nu$. Let $c^* = c^*(\mu, \nu)$ be the spreading speed given by Theorem 1.1, depending on the repartition of $\mu$ or $\nu$.

1. If $D \in \left[ 2d, d \left( 2 + \frac{\mu}{f'(0)} \right) \right]$, $\inf \{ c^*, \mu, \nu \in \Lambda_{\mu, \nu} \} = 2 \sqrt{df'(0)}$.

2. Fix $D > d \left( 2 + \frac{\mu}{f'(0)} \right)$, then $\inf \{ c^*, \mu, \nu \in \Lambda_{\mu, \nu} \} > 2 \sqrt{df'(0)}$.

Moreover, in both cases, minimizing sequences can be given by long range exchange terms of the form $\mu_R(y) = \frac{1}{R} \mu \left( \frac{y}{R} \right)$ or $\nu_R(y) = \frac{1}{R} \nu \left( \frac{y}{R} \right)$ with $R \to \infty$. 
Limit curve: infimum for the spreading speed

\[ \lambda^+ = \lambda^- = \lambda(c^*) \]

\( \Psi_1 \) is fixed. The extremal points of \( \Psi_2 \) do not depend on the repartition of \( \mu \) and \( \nu \).
First intermediate model

\[
\begin{aligned}
\frac{\partial_t u - D \partial_{xx} u}{\partial_t v - d \Delta v} &= -\bar{\mu} u + \bar{\nu} v(t, x, 0) & x \in \mathbb{R}, t > 0 \\
\partial_t v - d \Delta v &= f(v) + \mu(y) u(t, x) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\
v(t, x, 0^+) &= v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\
-d \left\{ \partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-) \right\} &= -\bar{\nu} v(t, x, 0) & x \in \mathbb{R}, t > 0.
\end{aligned}
\] (7)

- Exchanges field $\rightarrow$ road by boundary condition, *id est* $\nu = \bar{\nu} \delta_{y=0}$.
- Exchanges road $\rightarrow$ field by a function $\mu$ with nontrivial support.
- We get the same general results (existence, spreading, minimal speed, ...).
Maximum spreading speed

Parameters $d, D, f'(0), \bar{v}, \bar{\mu}$ are fixed. We consider the set of admissible exchanges

$$\Lambda_{\bar{\mu}} = \{ \mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \bar{\mu}, \mu \text{ even} \}.$$  

For $\mu \in \Lambda_{\bar{\mu}}$ there exists $c^*(\mu)$ spreading speed. Let $c_0^*$ be the BRR spreading speed.

**Proposition**

$$c_0^* = \sup \{ c^*(\mu), \mu \in \Lambda_{\bar{\mu}} \}.$$  

Fastest spreading for localised exchanges from the road to the field. The proof is an explicit computation.
Second intermediate model

\[
\begin{align*}
\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} &= -\overline{\mu} u + \int \nu(y) v(t, x, y) \, dy \quad x \in \mathbb{R}, \ t > 0 \\
\frac{\partial v}{\partial t} - d \Delta v &= f(v) - \nu(y) v(t, x, y) \quad (x, y) \in \mathbb{R} \times \mathbb{R}^*, \ t > 0 \\
v(t, x, 0^+) &= v(t, x, 0^-), \quad x \in \mathbb{R}, \ t > 0 \\
-d \left\{ \frac{\partial}{\partial y} v(t, x, 0^+) - \frac{\partial}{\partial y} v(t, x, 0^-) \right\} &= \overline{\mu} u(t, x) \quad x \in \mathbb{R}, \ t > 0
\end{align*}
\] (8)

- Exchanges field $\rightarrow$ road by function $\nu$ with nontrivial support.
- Exchanges road $\rightarrow$ field by boundary condition (\textit{id est} $\mu = \overline{\mu} \delta_0$).
- General theorems are preserved (existence, spreading, ...).

Do we get the same kind of results?
First case: selfsimilar exchanges

For fixed parameters \( d, D, f'(0), \mu, \nu \) we consider the set of admissible exchanges

\[
\Lambda_\nu = \{ \nu \in C_0(\mathbb{R}), \nu \geq 0, \int \nu = \nu, \nu \text{ even} \}.
\]

For a given function \( \nu \), we set

\[
\nu_\varepsilon(y) = \frac{1}{\varepsilon} \nu \left( \frac{y}{\varepsilon} \right) \implies c^*(\varepsilon).
\]

Proposition

Let us consider \( c^* \) as a function of the \( \varepsilon \) variable. Then there exists \( \varepsilon_0 \),

\[
\forall \varepsilon < \varepsilon_0, \quad c^*(\varepsilon) > c_0^*.
\]

- Localised exchanges seem to be a local minimizer for the spreading speed.
- It does not depend on the function \( \nu \).

Is it a general result, that is, are localised exchange terms a local minimizer for the spreading speed?
Second case: perturbation of a Dirac

Mixed exchanges field → road: boundary condition + small nonlocal contribution.

\[ \nu(y) = (1 - \varepsilon)\delta_0 + \varepsilon\nu(y), \quad \nu \in \Lambda_1 \rightarrow c^*(\varepsilon). \]

**Theorem**

There exist \( m_1 > 2 \) depending on \( f'(0) \), \( M_1 \) depending on \( \bar{\mu} \) such that:

1. If \( D < m_1 \) there exists \( \varepsilon_0 \) and \( \nu \in \Lambda_1 \) such that \( \forall \varepsilon < \varepsilon_0, c_0^* < c^*(\varepsilon) \);
2. if \( \bar{\mu} > 4 \) and \( D, f'(0) > M_1 \) there exists \( \varepsilon_0 \) such that \( \forall \nu \in \Lambda_1, \forall \varepsilon < \varepsilon_0, c_0^* > c^*(\varepsilon) \).

- No general result for this model.
- Various behaviours may happen even in the neighbourhood of localised exchanges.
The singular limit of concentrated exchanges

We consider two exchange functions \( \nu, \mu \). For all \( \varepsilon > 0 \) we define

\[
\nu_\varepsilon(y) = \frac{1}{\varepsilon} \nu \left( \frac{y}{\varepsilon} \right), \quad \mu_\varepsilon(y) = \frac{1}{\varepsilon} \mu \left( \frac{y}{\varepsilon} \right).
\]

- The exchange functions tend to Dirac measures \( \leftrightarrow \) boundary conditions.
- Formal convergence of the system to the initial BRR system.

For fixed parameters \( d, D, \bar{\mu}, f'(0) \) and initial conditions \((u_0, v_0)\) we have
- a spreading speed \( c^*_\varepsilon \);  
- a dynamical solution \((u_\varepsilon(t, x), v_\varepsilon(t, x, y))\);  
- a unique stationary solution \((U_\varepsilon, V_\varepsilon(y))\).

Question

Convergence of the dynamics as \( \varepsilon \to 0 \) ?
Uniform dynamics

We write \((u, v)\) the solution of the dynamical BRR system, \(c_0^*\) the spreading speed.

**Proposition**

- \(c_\varepsilon^* \to c_0^*\) as \(\varepsilon \to 0\), locally uniformly in all the parameters.
- \((U_\varepsilon, V_\varepsilon)\) tends \((\frac{\nu}{\mu}, 1)\), uniformly in \(y\).

**Theorem**

- Let \(c > c_0^*\). \(\forall \eta > 0\), \(\exists T_0, \varepsilon_0\) such that \(\forall t > T_0, \forall \varepsilon < \varepsilon_0\), \(\sup_{|x| > ct} |u_\varepsilon(x, t)| < \eta\).
- Let \(c < c_0^*\). \(\forall \eta > 0\), \(\exists T_0, \varepsilon_0\) such that \(\forall t > T_0, \forall \varepsilon < \varepsilon_0\), \(\sup_{|x| < ct} \left| u_\varepsilon(x, t) - \frac{\nu}{\mu} \right| < \eta\).
The previous theorem gives the commutation of the limits

$$\lim_{t \to \infty} \lim_{\varepsilon \to 0} u_{\varepsilon}(t, x + ct) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} u_{\varepsilon}(t, x + ct).$$

Main tool: convergence of the dynamical solutions.

**Theorem**

$$\|(u - u_{\varepsilon})(t)\|_{L^\infty(\mathbb{R})} + \|(v - v_{\varepsilon})(t)\|_{L^\infty(\mathbb{R}^2)} \quad \varepsilon \to 0 \quad \longrightarrow 0 \text{ locally uniformly in } t \in (0, +\infty).$$

Convergence local in time, global in space.

Idea of proof:

- Convergence of the linear operator.
- Duhamel’s formula, Gronwall type argument.

Conclusion

- Persistence of the initial results of Berestycki, Roquejoffre and Rossi.
- A new threshold for infinitely supported exchanges.
- Differences between the two exchange functions and their influence on the dynamics.
- Uniform dynamics for self-similar exchange terms.

Perspectives:
- Including reaction on the road: persistence of the differences ?
- Transition between classical and enhanced spreading for long range exchanges.
- More general kernels.
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Bistable entire solutions

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Problem under study

Bistable reaction-diffusion equation:

\[
\begin{cases}
    \partial_t u(t, x) - \Delta u(t, x) = f(u), & t \in \mathbb{R}, \ x \in \Omega, \\
    \partial_\nu u(t, x) = 0, & t \in \mathbb{R}, \ x \in \partial \Omega.
\end{cases}
\] (9)

The domain $\Omega$ is assumed to be a smooth infinite domain in the $x_1$–direction, i.e.

\[
\Omega = \left\{ (x_1, x'), x_1 \in \mathbb{R}, x' \in \omega(x_1) \subset \mathbb{R}^{N-1} \right\}.
\]

We also make a cylinder-like assumption:

\[
\omega(x_1) \xrightarrow{x_1 \to -\infty} \omega^\infty
\]

The domain is, in one direction at infinity, the straight cylinder $\mathbb{R} \times \omega^\infty$.

Figure: Example of asymptotically cylindrical domain
Assumptions:

- The reaction term $f$ is of bistable kind, with $\int_0^1 f(s) ds > 0$.
- The domain $\Omega$ is diffeomorphic to the cylinder $\Omega^\infty := \mathbb{R} \times \omega^\infty$.

Question

Existence and uniqueness of an entire (i.e. eternal) solution in such a domain connecting 0 to 1? Influence on the dynamics in such domains?
Influence of the geometry: some biological motivations

- **Population dynamics**
  - Population going through an isthmus
  - or a straight (fishes)

- **Cortical Spreading Depression**
  - CSD: transient and large depolarisation of the membrane of neurons. Propagation in the grey mater, absorption in the white matter.
  - Migraine with aura, stroke.
  - Blocking of CSD in rodent: inefficient in human.

These images are from the University of Wisconsin and Michigan State Comparative Mammalian Brain Collections, and from the National Museum of Health and Medicine.
Homogeneous case: the straight cylinder \( \mathbb{R} \times \omega^\infty \)

Travelling waves: solutions connecting 0 and 1 of the form

\[ u(t, x) = \varphi(x_1 - ct). \]

System for \((c, \varphi)\)

\[
\begin{align*}
-c\varphi' - \varphi'' &= f(\varphi) \\
\varphi(-\infty) &= 1, \quad \varphi(+\infty) = 0, \quad \varphi(0) = \theta.
\end{align*}
\]

Bistable nonlinearity: there exists a unique (up to translation) couple \((c, \varphi)\).

**Theorem**  

The travelling wave attracts the dynamics: if \(u_0\) is front-like, there exists \(x_0\) such that

\[
\sup_{t \rightarrow +\infty} |u(t, x) - \varphi(x_1 - ct + x_0)| \longrightarrow 0.
\]
Previous result

Under the assumption:

$$\Omega \cap \{x \in \mathbb{R}^N, x_1 < 0\} = \mathbb{R}^- \times \omega, \omega \subset \mathbb{R}^{N-1}.$$ 

The domain is equal to a cylinder in the left half space.

There exists a unique solution of (9) defined for all $t \in \mathbb{R}$ such that

$$u(t, x) - \phi(x_1 - ct) \quad \underset{t \to -\infty}{\rightarrow} \quad 0,$$ 

uniformly in $\Omega$.

This solution is increasing in time and converges to a steady state $u_\infty$.

- Blocking phenomenon by a narrow passage.
- Partial invasion in sufficiently large domain.
- Complete invasion in star-shaped domains and domains with decreasing cross-section.
Aim: get rid of the strong cylinder-like assumption. We suppose that $\omega(x_1)$ converges to $\omega^\infty$ at some exponential rate for the $C^{2,\alpha}$ topology.

**Theorem**

There exists a unique function $u(t, x)$ defined for $t \in \mathbb{R}$ and $x \in \Omega$ such that

$$\sup \{|u(t, x) - \varphi(x_1 - ct)|, x \in \Omega\} \underset{t \to -\infty}{\longrightarrow} 0.$$  

- The proof amounts to proving the stability of the bistable wave.
- All the other results (propagation, complete or partial, blocking) are preserved.
A case study

One dimensional problem:

\[ \partial_t u - \partial_{xx} u = f(u) (1 + g(x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R} \]

(10)

where \( g \) is a bounded perturbation that satisfies the assumption

there exists \( \kappa > 0, \ |g(x)| \leq e^{\kappa x} \) for all \( x \in \mathbb{R} \).

(11)

Notation: \( \rho_1 \) is the spectral gap of \( \mathcal{L} := -\partial_{xx} - c\partial_x - f'(\varphi) \).

Theorem

Set \( \varpi = \frac{\rho_1}{\|f'\|_{\infty}} \). If \( g \) satisfies (11) and \( g > -\varpi \), then there exists a function

\( u_\infty = u_\infty(t, x) \) defined for \( t \in \mathbb{R}, \ x \in \mathbb{R} \) solution of (10) which satisfies

\[ \|u_\infty(t, \cdot) - \varphi(\cdot - ct)\|_{L^\infty(\mathbb{R})} \xrightarrow{t \to -\infty} 0. \]

(12)

Remark: other results for this type of equation concerning transition fronts

A. Zlatoš

Idea of proof

For the case study:

1. long time stability of the wave under the perturbation;
2. compactness argument.

In a cylinder-like domain:

1. ideas of the case study;
2. estimate of the solution ahead of the front.
Conclusion and perspectives

- The spreading properties given by Berestycki-Bouhours-Chapuisat are preserved.
- We ask for an exponential convergence, but at an arbitrary rate. What about weaker convergence?
- In the case study, link with the theory of transition fronts.
- Time delay coming from the variation of the cross section.
Thank you for your attention.