Math 2263 - Quiz 6
Double integrals: polar coordinates and other things

Name: 
Score: 000000

This is a 20 minutes quiz, one-sided. No calculators, computers, cellphones, notes, book allowed. Show all work. No credit will be given for answers without work.

1. (5 points) Find the area of the part of the hyperbolic paraboloid with equation \( z = x^2 - y^2 \) that lies between the two cylinders \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

Solution: We want to find the area of the surface \( z(x, y) = x^2 - y^2 \) above the domain 
\[ D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \} . \]

The area is given by 
\[ A = \iint_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA . \]

Obviously \( \frac{\partial z}{\partial x} = 2x \), \( \frac{\partial z}{\partial y} = -2y \). We describe the domain in polar coordinates:
\[ D = \{(r, \theta) \mid 0 \leq \theta < 2\pi, 1 < r < 2 \} . \]

Hence we have:
\[ A = \iint_D \sqrt{1 + 4r^2 + 4} \, r \, dr \, d\theta = 2\pi \int_1^2 \sqrt{1 + 4r^2} \, r \, dr = \frac{\pi}{6} \left( 17\frac{3}{2} - 5\frac{3}{2} \right) . \]

2. We consider the lamina that occupies the domain \( D \), bounded by the two curves \( y = x \) and \( y = x^2 \). The density of the lamina is \( \rho(x, y) = y \).

(a) (4 points) Find the mass \( M \) of the lamina.

Solution: The domain is 
\[ D = \{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq x \} . \]

So the mass \( M \) is given by 
\[ M = \iint_D \rho(x, y) \, dA = \int_0^1 \int_{x^2}^{x} y \, dy \, dx = \int_0^1 \frac{y^2}{2} \bigg|_{x^2}^{x} \, dx = \int_0^1 \frac{1}{2} (x^2 - x^4) \, dx = \frac{1}{15} . \]

(b) (4 points) Compute the center of mass \( (\bar{x}, \bar{y}) \) of the lamina.

Solution: We have 
\[ \bar{x} = \frac{1}{M} \iint_D x \rho(x, y) \, dA, \quad \bar{y} = \frac{1}{M} \iint_D y \rho(x, y) \, dA . \]

We compute.
\[ \bar{x} = \frac{1}{M} \iint_D x \rho(x, y) \, dA = \frac{1}{M} \int_0^1 \int_{x^2}^{x} xy \, dy \, dx = \frac{1}{M} \int_0^1 \frac{x}{2} (x^2 - x^4) \, dx = \frac{15}{24} , \]
\[ \bar{y} = \frac{1}{M} \iint_D y \rho(x, y) \, dA = \frac{1}{M} \int_0^1 \int_{x^2}^{x} y^2 \, dy \, dx = \frac{1}{3M} \int_0^1 x^3 - x^6 \, dx = 5 \left( \frac{1}{4} - \frac{1}{7} \right) = \frac{15}{28} . \]