Math 2263 - Mid-term exam
Multiple integrals and other things

Name: Score: / 100 points

- This is a 50 minutes exam. No computers, cellphones, book allowed. A scientific calculator and a formula sheet (one page, single sided) is allowed.

- This exams contains 6 pages (including this cover page), plus 2 pages of scratch paper, and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Calculators may be used.

- Do not give numerical approximations to quantities such as \( \sin(5) \), \( \pi \), or \( \sqrt{2} \). However, you should simplify \( \cos\left(\frac{\pi}{2}\right) = 0 \), \( e^0 = 1 \), and so on.

The following rules apply:

- **Show your work, in a reasonably neat and coherent way**, in the space provided. **All answers must be justified** by valid mathematical reasoning supported by the results seen in class.

- A correct answer, unsupported by calculations, explanations, and/or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- Please circle your answers and underline the key steps in your explanations and computations.

- Quality of the writing, organization of your space, cleanliness, and writing with an actual pen are greatly appreciated by the grader.

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1. Let $E$ be the domain in $\mathbb{R}^3$ bounded from below by the half-cone $z = \sqrt{x^2 + y^2}$ and from above by the sphere $x^2 + y^2 + z^2 = 8$, i.e.

$$E = \{ (x, y, z), \ z \geq 0, \ x^2 + y^2 \leq z^2, \ x^2 + y^2 + z^2 \leq 8 \}.$$

(a) Give a parametrization of $E$ in cylindrical AND in spherical coordinates.

**Solution:** One way to parametrize the domain in cylindrical coordinates is to say that $z$ is between two surfaces, while $(r, \theta)$ describes some domain $D$ on the plane, that is $E = \{ (r, \theta, z) : (r, \theta) \in D, \ h_1(r, \theta) \leq z \leq h_2(r, \theta) \}$. The difficult part is to determine what is the domain $D$. It is the orthogonal projection of the solid onto the $xy$–plane. With a picture, we easily see that the boundary of this domain coincides with the intersection of the cone and the sphere. They intersect on the circle $x^2 + y^2 = 4$, hence $D = \{ x^2 + y^2 \leq 4 \}$. It reads that the cylindrical coordinates of $E$ are:

$$E = \{ (r, \theta, z) : 0 \leq \theta < 2\pi, \ 0 \leq r \leq 2, \ r \leq z \leq \sqrt{8 - r^2} \}.$$

The description in spherical coordinates are much more natural and given below.

(b) Choose the appropriate coordinate system to evaluate the triple integral

$$I = \iiint_E \sqrt{x^2 + y^2 + z^2} \, dV.$$

**Solution:** The shape of the domain as well as the form of the function is an invitation to use spherical coordinates. The parametrization of $E$ in spherical coordinates gives

$$E = \{ (\rho, \theta, \phi), \ 0 \leq \rho \leq 2\sqrt{2}, \ 0 \leq \phi \leq \frac{\pi}{4}, \ 0 \leq \theta \leq 2\pi \}.$$

We recall that $\rho = \sqrt{x^2 + y^2 + z^2}$ and $dV = \rho^2 \sin \phi \, d\rho d\theta d\phi$. Hence we have

$$I = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{2\sqrt{2}} \rho^3 \sin \phi \, d\rho d\theta d\phi.$$

Using Fubini’s theorem it yields

$$I = \int_0^{\pi/4} \sin \phi \, d\phi \int_0^{2\pi} \, d\theta \int_0^{2\sqrt{2}} \rho^3 \, d\rho$$

$$= - [\cos \phi]_0^{\pi/4} \cdot 2\pi \cdot \frac{1}{4} \left[ \rho^4 \right]_0^{2\sqrt{2}}$$

$$= 32\pi \left( 1 - \frac{\sqrt{2}}{2} \right).$$
2. Find the area of the part of the hyperbolic paraboloid with equation \( z = x^2 - y^2 \) that lies between the two cylinders \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

**Solution:** This problem is ex. 7 on 15.5, (part of your homework assignment), and was the first problem in a practice exam.

We want to find the area of the surface \( z(x, y) = x^2 - y^2 \) above the domain
\[
\mathcal{D} = \{ (x, y), \ 1 \leq x^2 + y^2 \leq 4 \}.
\]
The area is given by
\[
A = \int \int_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \ dA.
\]
Obviously \( \frac{\partial z}{\partial x} = 2x \), \( \frac{\partial z}{\partial y} = -2y \). We describe the domain in polar coordinates:
\[
\mathcal{D} = \{ (r, \theta), \ 0 \leq \theta < 2\pi, \ 1 < r < 2 \}.
\]
Hence we have:
\[
A = \int \int_D \sqrt{1 + 4x^2 + 4y^2} \ dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} \ r \ dr \ d\theta = 2\pi \int_1^2 r \sqrt{1 + 4r^2} \ dr
\]
\[
= 2\pi \left[ \frac{1}{12} (1 + 4r^2)^{\frac{3}{2}} \right]_1^2 = \frac{\pi}{6} \left( 17^{\frac{3}{2}} - 5^{\frac{3}{2}} \right)
\]
3. We consider the lamina that occupies the domain $\mathcal{D}$, bounded by the two curves $y = x$ and $y = x^2$. The density of the lamina is $\rho(x, y) = y$.

(a) Find the mass $M$ of the lamina.

**Solution:** The domain is

$$\mathcal{D} = \{(x, y), \ 0 \leq x \leq 1, \ x^2 \leq y \leq x\}.$$ 

So the mass $M$ is given by

$$M = \iint_\mathcal{D} \rho(x, y) dA = \int_0^1 \int_{x^2}^x ydydx = \int_0^1 \frac{y^2}{2} \bigg|_{x^2}^x \ dx = \int_0^1 \frac{1}{2} (x^2 - x^4) \ dx = \frac{1}{15}.$$

(b) Compute the center of mass $(\bar{x}, \bar{y})$ of the lamina.

**Solution:** We have

$$\bar{x} = \frac{1}{M} \iint_\mathcal{D} x \rho(x, y) dA, \quad \bar{y} = \frac{1}{M} \iint_\mathcal{D} y \rho(x, y) dA.$$ 

We compute.

$$\bar{x} = \frac{1}{M} \iint_\mathcal{D} x \rho(x, y) dA = \frac{1}{M} \int_0^1 \int_{x^2}^x xy \ dydx = \frac{1}{M} \int_0^1 \frac{x}{2} (x^2 - x^4) \ dx = \frac{15}{24}.$$

$$\bar{y} = \frac{1}{M} \iint_\mathcal{D} y \rho(x, y) dA = \frac{1}{M} \int_0^1 \int_{x^2}^x y^2 \ dydx$$

$$\quad \quad = \frac{1}{M} \int_0^1 \frac{y^3}{3} \bigg|_{x^2}^x \ dx = \frac{1}{3M} \int_0^1 x^3 - x^6 \ dx = 5 \left(\frac{1}{4} - \frac{1}{7}\right) = \frac{15}{28}.$$
4. Find the volume of the solid bounded by the two cones \( z^2 = x^2 + y^2 \) and \( z^2 = 16x^2 + 16y^2 \), and between the planes \( z = 0 \) and \( z = 2 \).

**Solution:** The description of the solid is an invitation to use cylindrical coordinates. For all \( z \) between 0 and 2, \((x,y)\) lie on the crown \( \frac{z^2}{16} \leq x^2 + y^2 \leq z^2 \), therefore a parametrization of the domain in cylindrical coordinates can be

\[
E = \{(r, \theta, z) : 0 \leq \theta < 2\pi, \quad 0 \leq z \leq 2, \quad \frac{z}{4} \leq r \leq z \}
\]

Its volume is

\[
V = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^2 \int_{z/4}^z r \, dr \, dz \, d\theta
\]

\[
= 2\pi \int_0^2 \left. \frac{r^2}{2} \right|_{z/4}^z dz = \pi \int_0^2 z^2 \left( 1 - \frac{1}{16} \right) dz
\]

\[
= \frac{15}{16} \pi \frac{z^3}{3} \bigg|_0^2 = \frac{5\pi}{2}.
\]
5. Give, if they exist, the extreme values of the function \( f(x, y, z) = xy^2z \) on the surface \( S := \{(x, y, z) : x^2 + y^2 + z^2 = 4\} \). Please carefully explain your mathematical reasoning. (What is the surface \( S \)?)

**Solution:** The surface \( S \) is the sphere of radius 2. So it is a closed and bounded surface, the function \( f \) is continuous (as a polynomial function), therefore it admits global maximum and minimum on the surface. Notice also that \( f \) admits both positive and negative values on the surface (for instance, \( f(2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3}) = 16/9 \) and \( f(-2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3}) = -16/9 \)). Therefore, 0 is NOT an extreme value. So, in our analysis, we will allow ourselves to divide by \( x, y, \) or \( z \), since \( f(0, y, z) = f(x, 0, z) = f(x, y, 0) = 0 \) is not an extreme value.

The surface \( S \) is the 0 level set of \( g(x, y, z) = x^2 + y^2 + z^2 - 4 \). The gradient of \( g \) is never equal to \( \mathbf{0} \) on \( S \), so a point of extremum must satisfy the Lagrange equation \( \nabla f = \lambda \nabla g \) for some \( \lambda \). It reads:

\[
\begin{align*}
  y^2z &= 2\lambda x \\
  2xyz &= 2\lambda y \\
  xy^2 &= 2\lambda z \\
  x^2 + y^2 + z^2 &= 4 
\end{align*}
\]

The second equation gives \( \lambda = xz \) (since \( y \neq 0 \) as explained above), and so from the first and third equation we deduce \( y^2 = 2x^2 \) (since \( z \neq 0 \)) and \( y^2 = 2z^2 \) (since \( x \neq 0 \)). Plugging these onto the equation of the sphere, (1) becomes

\[
\begin{align*}
  y^2 &= 2x^2 \\
  \lambda &= xz \\
  y^2 &= 2z^2 \\
  \frac{x^2}{2} + y^2 + \frac{y^2}{2} &= 4 
\end{align*}
\]

Solving the last equation in \( y \), we obtain two possible values, \( y = \sqrt{2} \) and \( y = -\sqrt{2} \). Solving in \( x \) and \( z \), the potential solution points are \((x, y, z) = (\pm 1, \pm \sqrt{2}, \pm 1)\). Notice that this gives 8 potential points. We evaluate the function \( f \) at these points, and get that:

- \( \max_S f = 2 \), obtained at the four points \((1, \sqrt{2}, 1), (1, -\sqrt{2}, -1), (-1, \sqrt{2}, -1), \) and \((-1, -\sqrt{2}, 1)\).
- \( \min_S f = -2 \), obtained at the four points \((-1, \sqrt{2}, 1), (1, -\sqrt{2}, 1), (1, \sqrt{2}, -1), \) and \((-1, -\sqrt{2}, -1)\).
Scratch paper