• This is a 50 minutes exam.
• No books, cell phones, calculators, etc. are allowed. One page of formula sheet, one sided, hand written, is allowed.
• This exam contains 6 pages (including this cover page), plus 2 pages of scratch paper, and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
• Do not give numerical approximations to quantities such as \( \sin(5) \), \( \pi \), or \( \sqrt{2} \). However, you should simplify \( \cos\left(\frac{\pi}{2}\right) = 0 \), \( e^0 = 1 \), and so on.

The following rules apply:

• **Show your work, in a reasonably neat and coherent way**, in the space provided. All answers must be justified by valid mathematical reasoning supported by the results seen in class.

• A correct answer, unsupported by calculations, explanations, and/or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

• Please circle your answers and underline the key steps in your explanations and computations.

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1. (20 points) Find a $2 \times 2$ matrix $A$ such that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of $A$ associated with the eigenvalue 5, and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A$ associated with the eigenvalue 10.

**Solution:** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such a matrix. The problem reads

$$A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 10 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$ 

It yields a linear system of 4 equations with 4 unknowns:

$$\begin{cases} 3a + b = 15 \\ 3c + d = 5 \\ a + 2b = 10 \\ c + 2d = 20, \end{cases}$$

which is easily solved in $a = 4, b = 3, c = -2, d = 11$. So, there is a unique matrix that satisfies the desired relations, given by

$$\begin{pmatrix} 4 & 3 \\ -2 & 11 \end{pmatrix}.$$
2. (20 points) Find the least square solution (or solutions) of the equation

\[ A \vec{x} = \vec{b} \]

where \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \vec{b} = \begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix} \).

**Solution:** The considered system is obviously inconsistent since the last equation reads \( 0 = 1 \). We consider the normal equation:

\[ A^\top A \vec{x}^* = A^\top \vec{b} \iff \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 7 \\ 6 \end{pmatrix}. \]

The normal equation has a unique solution, so the least square solution is given by

\[ \vec{x}^* = \begin{pmatrix} 7 \\ 6 \end{pmatrix}. \]
3. (20 points) We consider the following matrix \( A = \begin{pmatrix} 5 & 2 & -1 \\ -10 & -4 & 2 \\ -4 & -2 & 2 \end{pmatrix} \).

(a) (15 points) Determine if the matrix \( A \) is diagonalizable, and if it is give a diagonal matrix similar to \( A \) (you do NOT have to find the change of basis matrix).

**Solution:** We investigate the eigenvalues of the matrix \( A \). The characteristic polynomial is given by

\[
f_A(\lambda) = \begin{vmatrix} 5 - \lambda & 2 & -1 \\ -10 & -4 - \lambda & 2 \\ -4 & -2 & 2 - \lambda \end{vmatrix} = \begin{vmatrix} 5 - \lambda & 2 & -1 \\ -2\lambda & -\lambda & 0 \\ 1 - \lambda & 0 & 1 - \lambda \end{vmatrix}
\]

Hence, the matrix admits three eigenvalues, \( \lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2 \). All of them have algebraic multiplicities 1, hence their geometric multiplicities is also 1. It is possible to find an eigenbasis for the matrix \( A \), the matrix is diagonalizable.

It yields that the matrix \( A \) is similar to \( D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \).

(b) (5 points) What is the rank of the matrix \( A \)?

**Solution:** Since the geometric multiplicity of the eigenvalue 0 is 1, it yields that

\[
dim \ (ker \ A) = 1.
\]

With the rank nullity theorem, this implies that

\[
dim \ (Im \ A) = rank \ A = 2.
\]
4. (20 points) Let $V \subset \mathbb{R}^4$ be the linear subspace given by

$$V = \text{Span} \left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$ 

Find the orthogonal projection of the vector $9 \vec{e}_2 = \begin{pmatrix} 0 \\ 9 \\ 0 \\ 0 \end{pmatrix}$ onto the subspace $V$.

**Solution:** $V$ is a vector space of dimension 2, let us call $\vec{v}_1, \vec{v}_2$ the considered vectors. To find an explicit formula for the orthogonal projection onto $V$, we need to find an orthonormal basis $B = \{ \vec{u}_1, \vec{u}_2 \}$ of $V$. It appears that $\vec{v}_1 \cdot \vec{v}_2 = 0$, the two vectors are already orthogonal, we just have to normalize them. Hence we can choose

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}.$$

Notice that $\|\vec{v}_1\| = \|\vec{v}_2\| = \sqrt{9}$. Then the orthogonal projection is given by

$$proj_v \vec{w} = (\vec{w} \cdot \vec{u}_1)\vec{u}_1 + (\vec{w} \cdot \vec{u}_2)\vec{u}_2 = 2\vec{v}_1 + 2\vec{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 2 \\ 2 \end{pmatrix}.$$
5. (20 points) Consider the vector \( \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^4 \). Let

\[
W = \{ \vec{w} \in \mathbb{R}^4 : \vec{w} \text{ perpendicular to } \vec{v} \},
\]
subspace of all vectors perpendicular to \( \vec{v} \).

(a) (5 points) What is the dimension of \( W \)?

**Solution:** Let us recall that a vector \( \vec{w} \) is in \( W \) if and only if \( \vec{w} \cdot \vec{v} = 0 \). We have \( \text{dim } W = 3 \). Two ways to see it.

1. Let \( V = \text{Span}\{ \vec{v} \} \). Then, it is a simple observation that \( W = V^\perp \), so \( \text{dim } W = \text{dim } \mathbb{R}^4 - \text{dim } V = 4 - 1 = 3 \).

2. Consider the linear transformation

\[
T : \begin{cases} \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \\ \vec{x} \longmapsto \frac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2} \end{cases}
\]

\( T \) is a linear transformation, whose image is \( \mathbb{R} \), and whose kernel is \( W \). With the rank nullity theorem it yields

\[
\text{dim } W = \text{dim } \mathbb{R}^4 - \text{dim } \mathbb{R} = 4 - 1 = 3.
\]

(b) (15 points) Find a basis of \( W \).

**Solution:** We just need to find 3 linearly independent vectors in \( W \), i.e. three linearly independent vectors \( \vec{w}_1, \vec{w}_2, \vec{w}_3 \) such that \( \vec{v} \cdot \vec{w}_{1,2,3} = 0 \). It is a simple observation that

\[
\vec{w}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and } \vec{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
\]
for instance, work.