• This is a 50 minutes exam.

• No books, cell phones, calculator, etc. are allowed. One page of formula sheet, one sided, hand written, is allowed.

• This exams contains 7 pages (including this cover page), plus 2 pages of scratch paper, and 5 problems plus one bonus. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Calculators may be used.

• Do not give numerical approximations to quantities such as \( \sin(5) \), \( \pi \), or \( \sqrt{2} \). However, you should simplify \( \cos\left(\frac{\pi}{2}\right) = 0, \ e^0 = 1 \), and so on

The following rules apply:

• **Show your work, in a reasonably neat and coherent way**, in the space provided. All answers must be justified by valid mathematical reasoning. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.

• Mysterious or unsupported answers will not receive full credit. Your work should be mathematically correct and carefully and legibly written.

• A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

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**Total 100 pts**
1. (20 points) Are the following vectors in $\mathbb{R}^4$ linearly independent? Justify your answer.

\[
\begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 1 \\ 4 \end{pmatrix}.
\]
2. (20 points) We consider the $3 \times 3$ system, with unknowns $x, y, z$, associated to the following augmented matrix.

\[
\begin{bmatrix}
3 & 0 & k & 2 \\
3 & 2 & 3 & k \\
k & 2 & 3 & 3
\end{bmatrix}
\]

Depending on the parameter $k$, how many solutions does the system admit? Justify your answer.
3. (20 points) We consider the following subset of $\mathbb{R}^{2\times 2}$:

$$V = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

(a) (10 points) Prove that $V$ is a subspace of $\mathbb{R}^{2\times 2}$.

(b) (10 points) Give a basis of $V$. What is its dimension?
4. (20 points) We consider the transformation

\[ T : \begin{cases} \mathbb{R}^{2 \times 2} & \rightarrow & \mathbb{R}^{2 \times 2} \\ M & \mapsto & M \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \end{cases} \]

(a) (5 points) Prove that \( T \) is a linear transformation.

(b) (15 points) Give the matrix of \( T \) in the standard basis of \( \mathbb{R}^{2 \times 2} \).
5. (20 points) We consider the following subset of $\mathbb{R}^n$:

$$V := \left\{ \overrightarrow{x} \in \mathbb{R}^n : \sum_{k=0}^{n} x_k = x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

(a) (5 points) Prove that $V$ is a subspace of $\mathbb{R}^n$.

(b) (15 points) Find a basis for $V$. What is the dimension of $V$?
6. (10 points) This is a bonus exercise. To do only if you have finished everything else.

Let $W_1, W_2$ be two subspaces of a given vector space $V$. Prove that $W_1 \cup W_2$ is a subspace of $V$ if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. 
Scratch paper