Bistable entire solutions in cylinder-like domains

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Problem under study

Bistable reaction-diffusion equation:

\[
\begin{aligned}
\begin{cases}
\partial_t u(t, x) - \Delta u(t, x) = f(u), & t \in \mathbb{R}, \ x \in \Omega, \\
\partial_{\nu} u(t, x) = 0, & t \in \mathbb{R}, \ x \in \partial\Omega.
\end{cases}
\end{aligned}
\]

The domain \(\Omega\) is assumed to be a smooth infinite domain in the \(x_1\)-direction, i.e.

\[
\Omega = \left\{ (x_1, x'), x_1 \in \mathbb{R}, x' \in \omega(x_1) \subset \mathbb{R}^{N-1} \right\}.
\]

We also make a cylinder-like assumption:

\[
\omega(x_1) \xrightarrow{x_1 \to -\infty} \omega^\infty
\]

The domain is, in one direction at infinity, the straight cylinder \(\mathbb{R} \times \omega^\infty\).

Figure: Example of asymptotically cylindrical domain
Assumptions:

- The reaction term $f$ is of bistable kind, with $\int_0^1 f(s)ds > 0$.
- There exists a $C^{2\alpha}$ diffeomorphism that sends $\Omega$ onto $\mathbb{R} \times \omega^\infty = \Omega^\infty$.

Question

Existence and uniqueness of an entire (i.e. eternal) solution in such a domain connecting 0 to 1? Influence on the dynamics in such domains?
Plan

1 Introduction
   - The problem
   - Motivations
   - Homogeneous case
   - Previous and current results

2 A one dimensional case study

3 Proof of the theorem
Influence of the geometry: biological motivations

- **Population dynamics**
  - Population going through an isthmus
  - or a straight (fishes)

- **Cortical Spreading Depression**
  - CSD: transient and large depolarisation of the membrane of neurons. Propagation in the grey mater, absorption in the white matter.
  - Aura in migraine with aura.
  - Blocking of CSD in rodent: inefficient in human.

These images are from the University of Wisconsin and Michigan State Comparative Mammalian Brain Collections, and from the National Museum of Health and Medicine.
Homogeneous case

In a straight cylinder $\mathbb{R} \times \omega^\infty$, uniqueness (up to translation) of the solution given by the bistable wave:

$$u(t, x) = \varphi(x_1 - ct).$$

The couple $(c, \varphi)$ is the unique solution of

\begin{align*}
- c \varphi' - \varphi'' &= f(\varphi) \\
\varphi(-\infty) &= 1, \quad \varphi(+\infty) = 0, \quad \varphi(0) = \theta.
\end{align*}

The travelling wave attracts the dynamics: if $u_0$ is front-like, there exists $x_0$ such that

$$\sup |u(t, x) - \varphi(x_1 - ct + x_0)| \xrightarrow{t \to +\infty} 0.$$
Previous mathematical result (Berestycki-Bouhours-Chapuisat)
Under a strong cylinder-like assumption:

$$\Omega \cap \left\{ x \in \mathbb{R}^N, x_1 < 0 \right\} = \mathbb{R}^- \times \omega, \omega \in \mathbb{R}^{N-1}. $$

This assumption asserts that the domain is equal to a cylinder in the left half space.

There exists a unique solution of (1) defined for all $t \in \mathbb{R}$ such that

$$u(t, x) - \varphi(x_1 - ct) \rightarrow 0, \quad t \rightarrow -\infty,$$
uniformly in $\Omega$.

This solution is increasing in time and converges to a steady state $u_\infty$.

- Blocking phenomenon by a narrow passage.
- Partial invasion in sufficiently large domain.
- Complete invasion in star-shaped domains and domains with decreasing cross-section.

Figure: Two examples of domains $\Omega$
Aim: get rid of the strong cylinder-like assumption. We suppose that $\omega(x_1)$ converges to $\omega^\infty$ at some exponential rate for the $C^{2,\alpha}$ topology.

**Theorem (P.)**

There exists a unique function $u(t, x)$ defined for $t \in \mathbb{R}$ and $x \in \Omega$ such that

$$\sup \{|u(t, x) - \varphi(x_1 - ct)|, x \in \Omega\} \xrightarrow{t \to -\infty} 0.$$  

- The proof amounts to proving the stability of the bistable wave.
- All the other results (propagation, complete or partial, blocking) are preserved.
Case study

One dimensional problem:

\[ \partial_t u - \partial_{xx} u = f(u) (1 + g(x)) , \quad t \in \mathbb{R}, \ x \in \mathbb{R} \]  

(2)

where \( g \) is a bounded perturbation that satisfies the assumption

there exists \( \kappa > 0 \), \( |g(x)| \leq e^{\kappa x} \) for all \( x \in \mathbb{R} \).  

(3)

Theorem (P.)

There exists a positive constant \( \varpi \) which depends only on \( f \) such that if \( g \) satisfies (3) and \( g > -\varpi \), then there exists a function \( u_\infty = u_\infty (t, x) \) defined for \( t \in \mathbb{R}, \ x \in \mathbb{R} \) solution of (2) which satisfies

\[ \| u_\infty (t, .) - \varphi(\cdot - ct) \|_{L^\infty(\mathbb{R})} \longrightarrow 0. \]  

(4)

The constant \( \varpi \) is given by \( \varpi = \frac{\rho_1}{\| f' \|_\infty} \) where \( \rho_1 \) is the spectral gap of the linearised operator associated with the travelling wave.

Remark: other results for this type of equation concerning transition fronts (see Zlatoš).
Strategy of the proof (Berestycki, Hamel, Matano)

Define a sequence \((u_n)\) where the functions \(u_n\) are solutions of

\[
\begin{cases}
\partial_t u_n(t, x) - \Delta u_n(t, x) = f(u_n) & t > -n, x \in \mathbb{R} \\
u_n(-n, x) = \varphi(x_1 + cn).
\end{cases}
\]

(5)

- The sequence \((u_n)\) converges locally uniformly to an entire solution.
- In the strong cylinder-like assumption, control with sub and super-solutions.

Aim

Stability of the bistable wave.

Proposition

There exist \(N_0, K, \gamma\) such that for all \(n > N_0\), the solution \(u_n\) of (5) satisfies

\[
|u_n(t, x) - \varphi(x_1 - ct)| \leq Ke^{\gamma ct}, \quad \forall t \in [-n, -N_0[.
\]
Moving framework: $\xi = x_1 - ct + cn$. The equation becomes

$$\begin{cases}
\partial_t u - c\partial_\xi u - \partial_{\xi\xi} u = f(u)\left(1 + g(\xi + ct - cn)\right), \quad t > 0, \xi \in \mathbb{R} \\
u(0, \xi) = \varphi(\xi).
\end{cases}$$  \hspace{1cm} (6)

Spectral decomposition: consider the linear operator $\mathcal{L} := c\partial_\xi + \partial_{\xi\xi} + f'(\varphi)$. Then there exists $X_1 \simeq \mathcal{R}(\mathcal{L})$ such that

$$BUC(\mathbb{R}) = X_1 \oplus N(\mathcal{L}),$$

with $N(\mathcal{L}) = N(\mathcal{L}^2) = \varphi'\mathbb{R}$. Projection on $N(\mathcal{L})$:

$$P_\psi(\xi) = \langle e_*, \psi \rangle \varphi'(\xi) = \frac{1}{\Lambda} \left( \int_{\mathbb{R}} e^{cz} \varphi'(z) \psi(z) dz \right) \varphi'(\xi).$$

We denote $Q = I - P$ the projection on $\mathcal{R}(\mathcal{L})$. The operator $\mathcal{L}$ satisfies on $\mathcal{R}(\mathcal{L})$, for any $\rho < \rho_1$,

$$\left\| e^{t\mathcal{L}} \right\|_{\mathcal{R}(\mathcal{L})} \leq Ce^{-\rho t}.$$
Splitting of the problem

Lyapunov-Schmidt decomposition

There exists $\varepsilon$, as soon as $\|u(t) - \varphi\| \leq \varepsilon$, one can decompose

$$u(t, \xi) = \varphi(\xi + \chi(t)) + v(t, \xi)$$

with continuous $\chi$ and $\langle e_*, v(t) \rangle = 0$.

- A "linear" parabolic equation on $\mathcal{R}(\mathcal{L})$ and an ODE on $\mathcal{N}(\mathcal{L})$.
- Use the coercivity of $\mathcal{L}$ on $\mathcal{R}(\mathcal{L})$.
- Bootstrap argument.
Equation on $\mathcal{R}(\mathcal{L})$

\[
\begin{cases}
  v_t - \mathcal{L} v - (g(\xi + ct - cn)f'(\varphi)v - \langle e_*, gf'(\varphi)v \rangle \varphi') = Q [R + g(\xi + ct - cn)f(\varphi)] \\
  v(0, \xi) = 0
\end{cases}
\]

where $R = O(v^2 + \chi^2)$.

- If $n$ is large enough, $g(\xi + ct - cn)f(\varphi)$ is small.
- The "important" linear part is $\mathcal{L} + g(\xi + ct - cn)f'(\varphi)$.

$\implies$ coercive if $g > -\frac{\rho_1}{\|f'\|}$.

Formally, a Duhamel’s formula gives the estimate:

**Lemma 1**

There exist $\gamma > \alpha > 0$, as soon as $|v(t)|, |\chi(t)| < e^{\alpha(ct-cn)}$, then

$$
\|v(t)\|_\infty < Ce^{\gamma(ct-cn)}.
$$
Equation on $\mathcal{N}(\mathcal{L})$

\[
\begin{cases}
\chi'(t) = \langle e_*, R \rangle + \langle e_*, g(\xi + ct - cn)f'(\varphi) \rangle v + g(\xi + ct - cn)f(\varphi) \\
\chi(0) = 0.
\end{cases}
\]

- Once again, $g(\xi + ct - cn)f(\varphi)$ is small when $n$ is large.
- The second term controlled by lemma 1.

**Lemma 2**

As soon as $|\nu(t)|, |\chi(t)| < e^{\alpha(ct-cn)}$, 

\[
|\chi(t)| < C e^{\gamma(ct-cn)}.
\]

**Conclusion**

For large $n$, uniform control of $\|u_n(t) - \varphi\|_{L^\infty(\mathbb{R})}$. It gives the behaviour of the entire solution as $t \to -\infty$. 
In a 2-dimensional cylinder-like domain
To use the spectral decomposition, we need to be in a cylinder.

Assumption

There exists a $C^{2,\alpha}$ diffeomorphism

$$\Phi : \left\{ \begin{array}{c} \Omega \\ (x, y) \end{array} \right\} \mapsto \left\{ \begin{array}{c} \Omega^\infty \\ (x, z) = (x, \Phi(x, y)) \end{array} \right\}$$

such that there exist $\kappa, C > 0$, for all $M \in \mathbb{R}$,

$$\left\| \left( \Phi - Id \right) \right\|_{\Omega \cap \{ x < M \}, C^{2,\alpha}} \leq C \min\{ e^{\kappa M}, 1 \}.$$

Change of variables:

$$(\xi, z) = (x - ct, \Phi(x, y)),$$ $u(t, x, y) = \tilde{u}(t, \xi, z).$$
Equation in a straight cylinder

For all $n$, we study the stability of

\[
\begin{aligned}
&\tilde{u}_t - c\tilde{u}_\xi - \Delta\tilde{u} - f(\tilde{u}) - \tilde{u}_z\Delta\Phi - 2\tilde{u}_\xi\frac{\Phi_x}{\Phi_y} + \tilde{u}_{zz} (1 - \Phi_x^2 - \Phi_y^2) = 0 \quad t > 0, (\xi, z) \in \Omega^\infty \\
&\tilde{u}_z \left( \Phi_y + \frac{\Phi_x^2}{\Phi_y} \right) + \tilde{u}_\xi \frac{\Phi_x}{\Phi_y} = 0 \\
&\tilde{u}(0, \xi, z) = \varphi(\xi + cn).
\end{aligned}
\]  

(7)

- Equation in a cylinder.
- Some more linear terms, with small coefficients only for $x = \xi + ct \ll -1$.

**Lemma**

There exist $\alpha_1$ and a positive function $\psi$ such that

\[
(t, x, y) \mapsto \psi(y)e^{-\alpha_1(x-ct)}
\]

is a super-solution of the problem (1) on the set $\Omega \cap \{x > ct\}$.

Parabolic estimates: control of $\tilde{u}_\xi, \tilde{u}_z$ where $\Phi$ is large $\leadsto$ control of the linear terms.
Splitting of the problem

Lyapunov-Schmidt decomposition

There exists $\varepsilon$, as soon as $\|\tilde{u}(t) - \varphi(\cdot + cn)\| \leq \varepsilon$, one can decompose

$$\tilde{u}(t, \xi, z) = \varphi(\xi + cn + \chi(t, z)) + v(t, \xi, z)$$

with continuous $\chi$ and $\langle e_*, v(t, ., z) \rangle = 0$, for all $z \in \omega^\infty$.

The system becomes

$$\begin{cases} 
\chi_t \varphi' - \chi_{zz} \varphi' + v_t - L v - v_{zz} - R_1 = 0 & (\xi, z) \in \Omega^\infty \\
\chi_z \varphi' + v_z = R_2 & (\xi, z) \in \partial \Omega^\infty \\
\tilde{u}(0, \xi, z) = \varphi(\xi + cn) 
\end{cases}$$

(8)

with

$$\|R_1(t)\| + \|R_2(t)\| \leq C e^{\alpha_1(ct-cn)}$$

for large $n$, as soon as $|u - \varphi|$ remains "small."
Equation on $\mathcal{R}(\mathcal{L})$

\[
\begin{aligned}
\frac{\partial}{\partial t} v - \mathcal{L} v - \partial_{zz} v &= Q[R_1] & t > 0, (\xi, z) \in \Omega^\infty \\
\partial_z v &= Q[R_2] & t > 0, (\xi, z) \in \partial \Omega^\infty \\
v(0, \xi, z) &= 0.
\end{aligned}
\]  

(9)

The operators $\mathcal{L}$ and $\partial_{zz}$ commute; we can apply the Duhamel's formula. Coercivity of $e^{t\mathcal{L}}$ and boundedness of the heat operator gives

**Lemma 1**

There exist $\gamma > \beta > 0$, as soon as $\|v(t)\|_{C^2,\alpha} + \|\chi(t)\|_{C^2,\alpha} < e^{\beta(ct-cn)}$, then

$$\|v(t)\|_{C^2,\alpha} < Ce^{\gamma(ct-cn)}.$$
Equation on $\mathcal{N}(\mathcal{L})$

\[
\begin{cases}
\partial_t \chi - \partial_{zz} \chi = \langle e_*, R_2 \rangle, & t > 0, \, z \in \omega^\infty \\
\partial_z \chi = \langle e_*, R_3 \rangle, & t > 0, \, z \in \partial \omega^\infty \\
\chi(0, z) = 0.
\end{cases}
\] (10)

Once again, boundedness of the heat operator,

**Lemma 2**

There exist $\gamma > \beta > 0$, as soon as $\|v(t)\|_{C^2,\alpha} + \|\chi(t)\|_{C^2,\alpha} < e^{\beta(ct-cn)}$, then

\[\|\chi(t)\|_{C^2,\alpha} < Ce^{\gamma (ct-cn)}.\]

We can apply lemma 1 and lemma 2 in a bootstrap argument, which concludes the proof.
Remarks and questions

- Uniqueness follows easily using Berestycki-Hamel-Matano in a cylinder-like domain, not in the case study.
- The spreading properties given by Berestycki-Bouhours-Chapuisat are preserved.
- We ask for an exponential convergence, but at an arbitrary rate. What about weaker convergence?
- In the case study, transition front?
Thank you for your attention.