Large-time behavior of solutions of parabolic equations on the real line with convergent initial data

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Abstract

We consider the semilinear parabolic equation $u_t = u_{xx} + f(u)$ on the real line, where f is a locally Lipschitz function on $\mathbb{R}$. We prove that if a solution u of this equation is bounded and its initial value $u(x,0)$ has distinct limits at $x = \pm\infty$, then the solution is quasiconvergent, that is, all its limit profiles as $t \to \infty$ are steady states.

Key words: Parabolic equations on the real line, convergent initial data, quasiconvergence, convergence

1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where f is a locally Lipschitz function on $\mathbb{R}$ and $u_0 \in C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We denote by $u(\cdot, t, u_0)$ the unique classical solution of (1.1)-(1.2) and by $T(u_0) \in (0, +\infty]$ its maximal existence time. If u is bounded on $\mathbb{R} \times [0, T(u_0))$,

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then necessarily $T(u_0) = +\infty$, that is, the solution is global. In this paper, we are concerned with the behavior of bounded solutions as $t \to \infty$. A basic question we specifically want to address is whether, or to what extent, the large-time behavior of bounded solutions is governed by steady states of (1.1).

This question has long been settled for equation (1.1) considered on a bounded interval, instead of $\mathbb{R}$, and complemented by one of common boundary conditions, say Dirichlet, Neumann, Robin, or periodic. Namely, in that case each bounded solution converges, uniformly on the spatial interval, to a steady state [4, 15, 23]. In contrast, the large-time behavior of equation (1.1) on $\mathbb{R}$ is not generally so simple and is much less understood.

To talk about the behavior in more specific terms, recall that, by standard parabolic regularity estimates, any bounded solution of (1.1) has relatively compact orbit in $L^\infty_{loc}(\mathbb{R})$. In other words, any sequence $t_n \to \infty$ has a subsequence $\{t_{n_k}\}$ such that $u(\cdot, t_{n_k}) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$ for some continuous function φ. It is therefore natural to use the topology of $L^\infty_{loc}(\mathbb{R})$ when considering the convergence of solutions and related issues. Thus, we say that a bounded solution u is *convergent* if for some φ one has $u(\cdot, t) \to \varphi$ locally uniformly on $\mathbb{R}$. Of course, the convergence may take place in stronger topologies, but we take the convergence in $L^\infty_{loc}(\mathbb{R})$, the topology in which the orbit is compact, as a natural minimal requirement.

While the convergence of the solution of (1.1), (1.2) has been proved under various conditions on u_0 and f [2, 6, 7, 9, 10, 11, 12, 16, 17, 20, 24], it is not the general behavior of bounded solutions even when $f \equiv 0$, that is, when (1.1) is the linear heat equation. As observed in [5], if u_0 takes values 0 and 1 on suitably spaced long intervals with sharp transitions between them, then, as $t \to \infty$, $u(\cdot, t)$ approaches 0 along a sequence of times $t_n \to \infty$ and 1 along another such sequence (the convergence is in $L^\infty_{loc}(\mathbb{R})$ in both cases).

As we explain shortly, for the linear equation the large-time behavior of any bounded solution is still governed by steady states in the sense that every limit profile of any such solution is a steady state. Here a *limit profile* of a bounded solution u of (1.1) refers to any element of the ω-limit set of u:

$$\omega(u) := \{\varphi \in C_b(\mathbb{R}) : \ u(\cdot, t_n) \to \varphi \text{ for some sequence} t_n \to \infty\}, \qquad (1.3)$$

where the convergence is in $L^\infty_{loc}(\mathbb{R})$. If the solution u corresponds to a given initial datum u_0, we also write $\omega(u_0)$ for $\omega(u)$. We say that a bounded solution u of (1.1) is *quasiconvergent* if $\omega(u)$ consists entirely of steady states. Thus, a quasiconvergent solution approaches a set of steady states, from which it follows that $u_t(\cdot,t) \to 0$, locally uniformly on $\mathbb{R}$, as $t \to \infty$. This makes quasiconvergent solutions hard to distinguish—numerically, for example—from convergent solutions; they move very slowly at large times.

In the case of the linear heat equation, the quasiconvergence of each bounded solution follows from the invariance property of the ω-limit set: $\omega(u)$ consists of *entire* solutions of (1.1), by which we mean solutions defined for all $t \in \mathbb{R}$. If u is bounded, then the entire solutions in $\omega(u)$ are bounded as well and, by the Liouville theorem for the linear heat equation, all such solutions are constant.

In nonlinear equations, a common way to prove the quasiconvergence of a solution is by means of a Lyapunov functional. For equation (1.1), the following energy functional is used frequently:

$$E(v) := \int_{-\infty}^{\infty} \big(\frac{v_x^2(x)}{2} - F(v(x)) \big)\, dx, \qquad F(v) := \int_0^v f(s)\, ds. \tag{1.4}$$

Of course, for this functional to be defined along a solution, one needs assumptions on f and u; but when such assumptions are made, it can be shown that $t \mapsto E(u(\cdot,t))$ is nonincreasing and consequently u is quasiconvergent (see, for example, [11] for results of this form).

For solutions which are not assumed to be bounded in an integral norm, the energy E is usually not very useful.[1] In fact, bounded solutions of nonlinear equations (1.1) are not quasiconvergent in general. The existence of non-quasiconvergent solutions for some equations of the form (1.1) was strongly indicated by results of [8]. It was later demonstrated by various examples in [19, 20]. Moreover, the results of [19, 20] show that non-quasiconvergent bounded solutions occur quite "frequently" in (1.1). They exist whenever there is an interval $[a,b]$ on which f is bistable: $f(a) = f(b) = 0$, $f'(a), f'(b) < 0$, and there is $\gamma \in (a,b)$ such that $f < 0$ in (a,γ) and $f > 0$ in (γ, b). This is clearly a robust class of nonlinearities.

On the other hand, several classes of initial data u_0 have been identified for which the solutions are quasiconvergent, if bounded. These include nonnegative localized data, or, nonnegative elements of $C_0(\mathbb{R})$, in the case $f(0) = 0$ [16]; as well as front-like initial data, by which we mean functions

[1] Note, however, that [13, 14] made a good use of (1.4) with the integral taken over the intervals $(-R,R)$, $R \gg 1$, instead of $(-\infty,\infty)$. As proved in [13], the ω-limit set of each bounded solution contains a steady state.

$u_0 \in C(\mathbb{R})$ satisfying $a \leq u_0 \leq b$, $u_0(-\infty) = b$, $u_0(\infty) = a$ for some zeros $a < b$ of f [21] (see [18] for a more detailed overview of quasiconvergence and related results). Here and below, $C_0(\mathbb{R})$ stands for the space of all continuous functions on $\mathbb{R}$ converging to 0 at $x = \pm\infty$. Note that the sign restriction in the case of localized initial data is essential; examples of non-quasiconvergent solutions [19, 20] do include some with sign-changing initial data in $C_0(\mathbb{R})$.

In this paper, we consider a class of initial data with includes in particular all front-like initial data, but without any sign restrictions like $a \leq u_0 \leq b$. Namely, we consider initial data u_0 in the space

$$\mathcal{V} := \{v \in C_b(\mathbb{R}) : \text{ the limits } v(-\infty),\, v(+\infty) \in \mathbb{R} \text{ exist}\}\,. \tag{1.5}$$

Note that the property of having finite limits at $\pm\infty$ is preserved by the solutions of (1.1), (1.2): if $u_0(\pm\infty)$ exist, then $u(\pm\infty, t, u_0)$ exist for all $t \in (0, T(u_0))$ (these limits vary with t in general, see Lemma 3.1 below for a more precise statement). This means that the space $\mathcal{V}$ is an invariant space for (1.1), just like the space $C_b(\mathbb{R})$, or the space $C_0(\mathbb{R})$ in the case $f(0) = 0$. Since $\mathcal{V}$ is a closed subspace of $C_b(\mathbb{R})$, it is a Banach space when equipped with the supremum norm.

Of course, $\mathcal{V}$ contains functions u_0 with $u_0(-\infty) = u_0(\infty)$—in particular, it contains $C_0(\mathbb{R})$—so we do not have quasiconvergence of all bounded solutions with initial data $u_0 \in \mathcal{V}$. As it turns out, however, $u_0(-\infty) = u_0(\infty)$ is the only case when the quasiconvergence may fail to hold. This is a part of our main theorem, which we state precisely after introducing some notation and terminology.

Consider the ordinary differential equation for the steady states of (1.1):

$$u_{xx} + f(u) = 0, \qquad x \in \mathbb{R}. \tag{1.6}$$

The corresponding first-order system,

$$u_x = v, \qquad v_x = -f(u), \tag{1.7}$$

is a Hamiltonian system, which has only four types of bounded orbits: equilibria, nonstationary periodic orbits (or, closed orbits), homoclinic orbits, and heteroclinic orbits. We adopt the following common terminology concerning steady states φ of (1.1). We say φ is a *ground state* of (1.6) if the orbit of (φ, φ') is a homoclinic orbit of (1.7); and φ is a *standing wave* of (1.1) if the orbit of (φ, φ') is a homoclinic orbit of (1.7).

Under our standing hypothesis that f is locally Lipschitz on $\mathbb{R}$, we have the following result.

Theorem 1.1. *Assume that $u_0 \in \mathcal{V}$ and $u_0(-\infty) \neq u_0(\infty)$. If the solution $u(\cdot,\cdot,u_0)$ of* (1.1), (1.2) *is bounded, then it is quasiconvergent: $\omega(u_0)$ consists entirely of steady states of* (1.1). *More specifically, if $\varphi \in \omega(u_0)$, then it is a constant steady, or a ground state of* (1.6), *or a standing wave of* (1.1).

Remark 1.2. (i) An even more precise description of $\omega(u_0)$ will come out of the proof of the theorem. Namely, consider the following possibility:

(M) u is eventually monotone in space: given any $k \in \mathbb{R}$, one has $u_x(x,t) \neq 0$ for all $x \in (-k,k)$ if t is sufficiently large.

If (M) holds, we will prove that u is quasiconvergent and each element of $\omega(u_0)$ is a constant steady state or a standing wave of (1.1). If (M) does not hold, we show that u is even convergent, with $\omega(u_0) = \{\varphi\}$, where φ is a constant steady or a ground state of (1.6).

(ii) Theorem 1.1 in particular shows that nonconstant periodic steady states are never elements of $\omega(u_0)$ for $u_0 \in \mathcal{V}$.

(iii) Clearly, the set of all functions $u_0 \in \mathcal{V}$ satisfying $u_0(-\infty) \neq u_0(+\infty)$ is open and dense in $\mathcal{V}$ (with the supremum norm). Thus, Theorem 1.1 has an interesting additional feature in that it shows that quasiconvergence is generic in $\mathcal{V}$: the solution of (1.1) is quasiconvergent, if bounded, for an open and dense set of initial data in $\mathcal{V}$. In contrast, using the constructions from [19], one can show that this genericity statement is not valid if one replaces $\mathcal{V}$ with $C_b(\mathbb{R})$ or $C_0(\mathbb{R})$.

Theorem 1.1 is proved in Section 3. Several preliminary results concerning steady states, zero number, and ω-limit sets that are needed for the proof are recalled in Section 2.

As Theorem 1.1 concerns bounded solutions only, modifying f outside the range of the solution, we may assume without loss of generality that f satisfies the following condition:

$$\text{there exists } \kappa > 0 \text{ such that for all } |u| > \kappa \text{ one has } f(u) = \frac{u}{2}. \tag{1.8}$$

This will be convenient in the next section.

2 Preliminaries

2.1 Steady states and their trajectories in the phase plane

In this subsection, we recall several technical results concerning the steady states of (1.1), or, solutions of (1.6). The first-order system (1.7) corresponding to (1.6) is Hamiltonian with respect to the energy

$$H(u,v)=\frac{v^2}{2}+F(u), \tag{2.1}$$

where $F(u)=\displaystyle\int_0^u f(s)ds$. Thus, each orbit of (1.7) is contained in a level set of H. The level sets are symmetric with respect to the $v-$axis, and our extra hypothesis (1.8) implies that they are all bounded. Therefore, all orbits of (1.7) are bounded and, as already mentioned in the introduction, there are only four types of them: equilibria (all of which are on the $u-$axis), non-stationary periodic orbits, homoclinic orbits (corresponding to ground states of (1.6)), and heteroclinic orbits (corresponding to standing waves of (1.1)).

Each non-stationary periodic orbit $\mathcal{O}$ is symmetric about the $u-$axis and for some $p<q$ one has

$$\begin{aligned}&\mathcal{O}\cap\{(u,0):u\in\mathbb{R}\}=\{(p,0),(q,0)\}\\&\mathcal{O}\cap\{(u,v):v>0\}=\left\{\left(u,\sqrt{2(F(p)-F(u))}\right):u\in(p,q)\right\}.\end{aligned} \tag{2.2}$$

The following result of [16] gives a description of the phase plane portraits of (1.7) with all the periodic orbits removed. Let

$$\begin{aligned}\mathcal{E}&:=\{(a,0):f(a)=0\}\ \text{(the set of all equilibria of (1.7))},\\\mathcal{P}_0&:=\{(a,b)\in\mathbb{R}^2:(a,b)\text{ lies on a non-stationary periodic orbit of (1.7)}\}\\\mathcal{P}&:=\mathcal{P}_0\cup\mathcal{E}.\end{aligned}$$

Lemma 2.1. [16, Lemma 3.1] *The following two statements are valid.*

(i) Let Σ be a connected component of $\mathbb{R}^2\setminus\mathcal{P}_0$. Then Σ is a compact set contained in a level set of the Hamiltonian H and one has

$$\Sigma=\left\{(u,v)\in\mathbb{R}^2:u\in J,\ v=\pm\sqrt{2(c-F(u))}\right\}$$

where c is the value of H on Σ and $J = [p, q]$ for some $p, q \in \mathbb{R}$ with $p \leq q$. Moreover, if $(u, 0) \in \Sigma$ and $p < u < q$, then $(u, 0)$ is an equilibrium. The point $(p, 0)$ is an equilibrium or it lies on a homoclinic orbit; the same is true for the point $(q, 0)$.

(ii) Each connected component of the set $\mathbb{R}^2 \setminus \mathcal{P}$ consists of a single orbit of (1.7), either a homoclinic orbit or a heteroclinic orbit.

The following lemma is a simple consequence of the continuity of the solutions of (1.7) with respect to the initial conditions.

Lemma 2.2. *Let $\mathcal{O}_n$, $n = 1, 2, \dots$ be a sequence of non-stationary periodic orbits of (1.7) with the minimal periods ρ_n, $n = 1, 2, \dots$, respectively. Suppose that for some bounded set $K \subset \mathbb{R}^2 \setminus \mathcal{P}$ one has dist$(\mathcal{O}_n, K) \underset{n\to\infty}{\longrightarrow} 0$. Then $\rho_n \underset{n\to\infty}{\longrightarrow} \infty$.*

If φ is a C^1 bounded function on $\mathbb{R}$, we let

$$\tau(\varphi) := \{(\varphi(x), \varphi_x(x)) : x \in \mathbb{R}\}$$

and refer to this set as the *spatial trajectory (or orbit)* of φ. If φ is a solution of (1.6), then $\tau(\varphi)$ is the usual orbit of the solution (φ, φ_x) of (1.7).

2.2 Invariance of the ω-limit set

Recall that the $\omega-$limit set of a bounded solution u of (1.1), denoted by $\omega(u)$, or $\omega(u_0)$ if the initial value of u is given, is defined as in (1.3), with the convergence in $L^\infty_{loc}(\mathbb{R})$. By standard parabolic estimates the trajectory $\{u(\cdot, t),\ t \geq 1\}$ of u is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that $\omega(u)$ is nonempty, compact, and connected in (the metric space) $L^\infty_{loc}(\mathbb{R})$ and it attracts the solution in the following sense:

$$\mathrm{dist}_{L^\infty_{loc}(\mathbb{R})}\left(u(\cdot, t), \omega(u)\right) \underset{t\to\infty}{\longrightarrow} 0. \tag{2.3}$$

It is also a standard observation that if $\varphi \in \omega(u)$, there exists an entire solution $U(x, t)$ of (1.1) such that

$$U(\cdot, 0) = \varphi, \qquad U(\cdot, t) \in \omega(u) \quad (t \in \mathbb{R}). \tag{2.4}$$

Here, *an entire solution* of (1.1) refers to a solution defined for all $x \in \mathbb{R}$, $t \in \mathbb{R}$. Let us briefly recall how such an entire solution U is found. By parabolic regularity estimates, u_t, u_x, u_{xx} are bounded on $\mathbb{R} \times [1, \infty)$ and are globally $\alpha-$Hölder for any $\alpha \in (0, 1)$. If $u(\cdot, t_n) \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^\infty_{loc}(\mathbb{R})$ for some $t_n \to \infty$, we consider the sequence $u_n(x, t) := u(x, t + t_n)$, $n = 1, 2 \dots$. Passing to a subsequence if necessary, we have $u_n \to U$ in $C^1_{loc}(\mathbb{R}^2)$ for some function U; this function U is then easily shown to be an entire solution of (1.1). By definition, U satisfies (2.4). Note that the entire solution U is determined uniquely by φ; this follows from the uniqueness and backward uniqueness for the Cauchy problem (1.1), (1.2).

Using similar compactness arguments, one shows easily that $\omega(u)$ is connected in $C^1_{loc}(\mathbb{R})$. Hence, the set

$$\{(\varphi(x), \varphi_x(x)) : \varphi \in \omega(u), x \in \mathbb{R}\} = \bigcup_{\varphi\in\omega(u)} \tau(\varphi)$$

is connected in $\mathbb{R}^2$. Also, obviously, $\tau(\varphi)$ is connected in $\mathbb{R}^2$ for all $\varphi \in \omega(u)$.

We will also use the following result (see [16, Lemma 4.3] or [21, Lemma 6.10] for a proof).

Lemma 2.3. *Let u be a bounded solution of* (1.1). *If $\varphi \in \omega(u)$, ψ is a solution of (1.6), and $\tau(\varphi) \subset \tau(\psi)$, then φ is a shift of ψ.*

2.3 Zero number for linear parabolic equations

In this subsection, we consider solutions of a linear parabolic equation

$$v_t = v_{xx} + c(x, t)v, \qquad x \in \mathbb{R},\ t \in (s, T)\,, \tag{2.5}$$

where $-\infty \le s < T \le \infty$ and c is a bounded measurable function. For an interval $I = (a, b)$, with $-\infty \le a < b \le \infty$, we denote by $z_I(v(\cdot, t))$ the number, possibly infinite, of zeros $x \in I$ of the function $x \mapsto v(x, t)$. If $I = \mathbb{R}$ we usually omit the subscript $\mathbb{R}$:

$$z(v(\cdot, t)) := z_{\mathbb{R}}(v(\cdot, t)).$$

The following intersection-comparison principle holds [1, 3].

Lemma 2.4. *Let v be a nontrivial solution of* (2.5) *and $I = (a, b)$, with $-\infty \le a < b \le \infty$. Assume that the following conditions are satisfied:*

- *if $b < \infty$, then $v(b, t) \neq 0$ for all $t \in (s, T)$,*

- *if $a > -\infty$, then $v(a,t) \neq 0$ for all $t \in (s,T)$.*

Then the following statements hold true.

(i) For each $t \in (s,T)$, all zeros of $v(\cdot,t)$ are isolated. In particular, if I is bounded, then $z_I(v(\cdot,t)) < \infty$ for all $t \in (s,T)$.

(ii) The function $t \mapsto z_I(v(\cdot,t))$ is monotone non-increasing on (s,T) with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.

(iii) If for some $t_0 \in (s,T)$ the function $v(\cdot,t_0)$ has a multiple zero in I and $z_I(v(\cdot,t_0)) < \infty$, then for any $t_1, t_2 \in (s,T)$ with $t_1 < t_0 < t_2$, one has

$$z_I(v(\cdot,t_1)) > z_I(v(\cdot,t_0)) \geq z_I(v(\cdot,t_2)). \tag{2.6}$$

If (2.6) holds, we say that $z_I(v(\cdot,t))$ drops in the interval (t_1,t_2).

Remark 2.5. It is clear that if the assumptions of Lemma 2.4 are satisfied and for some $t_0 \in (s,T)$ one has $z_I(v(\cdot,t_0)) < \infty$, then $z_I(v(\cdot,t))$ can drop at most finitely many times in (t_0,T); and if it is constant on (t_0,T), then $v(\cdot,t)$ has only simple zeros in I for all $t \in (t_0,T)$. In particular, if $T = \infty$, there exists $t_1 < \infty$ such that $t \mapsto z_I(v(\cdot,t))$ is constant on (t_1,∞) and all zeros are simple.

Using the previous remark and the implicit function theorem, we obtain the following corollary.

Corollary 2.6. *Assume that the assumptions of Lemma 2.4 are satisfied and that the function $t \mapsto z_I(v(\cdot,t))$ is constant on (s,T). If for some $(x_0,t_0) \in I \times (s,T)$ one has $v(x_0,t_0) = 0$, then there exists a C^1- function $t \mapsto \eta(t)$ defined for $t \in (s,T)$ such that $\eta(t_0) = x_0$ and $v(\eta(t),t) = 0$ for all $t \in (s,T)$.*

We will also need the following robustness lemma (see [6, Lemma 2.6]).

Lemma 2.7. *Let $w_n(x,t)$ be a sequence of functions converging to $w(x,t)$ in $C^1(I \times (s,T))$ where I is an open interval. Assume that $w(x,t)$ solves a linear equation (2.5), $w \not\equiv 0$, and $w(\cdot,t)$ has a multiple zero $x_0 \in I$ for some $t_0 \in (s,T)$. Then there exist sequences $x_n \to x_0$, $t_n \to t_0$ such that for all sufficiently large n the function $w_n(\cdot,t_n)$ has a multiple zero at x_n.*

In the next section we frequently use the following standard facts, often without notice. If u, $\bar{u}$ are bounded solutions of the nonlinear equation (1.1) with a Lipschitz nonlinearity, then their difference $v = u - \bar{u}$ satisfies a linear equation (2.5) with some bounded measurable function c. Similarly, $v = u_x$ and $v = u_t$ are solutions of such a linear equation.

3 Proof of the main result

Throughout this section we assume the hypotheses of Theorem 1.1 to be satisfied: $u_0 \in \mathcal{V}$ (cp. (1.5)) and

$$\alpha := u_0(-\infty) \neq \beta := u_0(+\infty). \tag{3.1}$$

Further, we assume that the solution $u(x,t)$ of (1.1)-(1.2) is bounded.

We denote

$$\theta_-(t) := \lim_{x\to-\infty} u(x,t), \qquad \theta_+(t) := \lim_{x\to\infty} u(x,t). \tag{3.2}$$

These limits exist according to the following lemma (the proof can be found in [22, Theorem 5.5.2], for example).

Lemma 3.1. *The limits $\theta_-(t), \theta_+(t)$ exist for all $t > 0$ and are solutions of the following initial-value problems:*

$$\dot{\theta}_\pm = f(\theta_\pm), \qquad \theta_-(0) = \alpha,\ \theta_+(0) = \beta. \tag{3.3}$$

3.1 The reflection principle and stabilization of the critical points

We employ the reflection-invariance of equation (1.1) in a way quite common in studies of spatially homogeneous parabolic equations. For any $\lambda \in \mathbb{R}$, consider the function $V_\lambda u$ defined by

$$V_\lambda u(x,t) = u(2\lambda - x, t) - u(x,t), \quad x \in \mathbb{R},\, t \geq 0. \tag{3.4}$$

Being the difference of two solutions of (1.1), $V_\lambda u$ is a solution of the linear equation (2.5) for some bounded function c.

We apply zero-number results to the functions $V_\lambda u$, $\lambda \in \mathbb{R}$. First observe that for any $\lambda \in \mathbb{R}$, hypothesis (3.1) and Lemma 3.1 imply that for $t \geq 0$ the function $V_\lambda u(x,t)$ has the limits as $x \to \pm\infty$ given by $\pm(\theta^+(t) - \theta^-(t))$, and these limits are both nonzero for all sufficiently small $t > 0$. Therefore, by Lemma 2.4, $z(V_\lambda u(\cdot,t))$ is finite for all $t > 0$. By Remark 2.5, there is $T = T(\lambda)$ such that the function $t \mapsto z(V_\lambda u(\cdot,t))$ is constant on $(T(\lambda),\infty)$ and, for all $t > T(\lambda)$, all zeros of $V_\lambda u(\cdot,t)$ are simple. In particular, since $x = \lambda$ is always a zero of $V_\lambda u$ by the definition of $V_\lambda u$, we have

$$-2u_x(\lambda,t) = \partial_x V_\lambda u(x,t)|_{x=\lambda} \neq 0 \quad (t > T(\lambda)). \tag{3.5}$$

We use this to prove the following result (a similar theorem for solutions periodic in space can be found in [4]).

Proposition 3.2. *For any open bounded interval $I \subset \mathbb{R}$, there exist $T_1 = T_1(I) > 0$ and an integer $N \geq 0$ such that, for all $t > T_1$, the function $u_x(\cdot, t)$ has exactly N zeros in I, all of them simple. Moreover, if $N > 0$ and $\eta_1(t) < \dots \eta_N(t)$ denote the zeros of $u_x(\cdot, t)$ in I for $t > T_1$, then the functions $\eta_i(t)$, $i = 1, \dots, N$, are of class C^1, and for some $x_i^\infty \in \overline{I}$ one has*

$$\eta_i(t) \underset{t\to\infty}{\longrightarrow} x_i^\infty \quad (i = 1, \dots, N). \tag{3.6}$$

Proof. Let $I = (a, b)$ with $a, b \in \mathbb{R}$ be any open bounded interval. Applying (3.5) to $\lambda \in \{a, b\}$ and setting $T_0 := \max(T(a), T(b))$, we obtain

$$u_x(a, t) \neq 0, \ u_x(b, t) \neq 0 \quad (t > T_0).$$

The function $u_x(x, t)$ is a solution of a linear equation (2.5). Hence, by Lemma 2.4, there exist $T_1 \geq T_0$ and $N \geq 0$ such that for all $t > T_1$ the function $u_x(\cdot, t)$ has exactly N zeros in I, all of them simple. If $N = 0$, the proof of Proposition 3.2 is finished.

Assume that $N \geq 1$. Let $\eta_1(t) < \dots \eta_N(t)$ the zeros of $u_x(\cdot, t)$ in I. The simplicity of these zeros and the implicit function theorem imply that the functions $\eta_i(t)$, $i = 1, \dots, N$, are of class C^1. It remains to show that these functions are all convergent. Assume for a contradiction that for some $i \in \{1, \dots, N\}$ the function $\eta_i(t)$ is not convergent as $t \to \infty$. Then it admits at least two accumulation points $x_i^\infty \neq \tilde{x}_i^\infty$. Consequently, for $\lambda := (x_i^\infty + \tilde{x}_i^\infty)/2$ there is a sequence $t_n \to \infty$ such that $\eta_i(t_n) = \lambda$ for all n, which contradicts (3.5). $\square$

We complement the previous results with the following useful information.

Lemma 3.3. *Under the notation of Proposition 3.2, if $N > 0$, then for any $\lambda \in \{x_i^\infty : i = 1, \dots, N\}$ one has*

$$V_\lambda u(\cdot, t) \underset{t\to\infty}{\longrightarrow} 0 \quad \text{in } C^1_{loc}(\mathbb{R}). \tag{3.7}$$

Proof. It is sufficient to prove the following statement. Given any sequence $t_n \to \infty$, one can pass to a subsequence such that (3.7) holds with t replaced by t_n.

This will be shown using an entire solution U, as constructed in Section 2.2. Passing to a subsequence of $\{t_n\}$, we may assume that $u(\cdot, \cdot+t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$, where U is an entire solution of (1.1). Then also $V_\lambda u(\cdot, \cdot+t_n) \to V_\lambda U$ in $C^1_{loc}(\mathbb{R}^2)$, for any λ. If now $\lambda \in \{x_i^\infty : i = 1, \dots, N\}$, then

$$\partial_x V_\lambda u(x,t) = -2u_x(\lambda, t) \underset{t\to\infty}{\longrightarrow} 0.$$

It follows that

$$\partial_x V_\lambda U(\lambda, t) = 0 \quad (t \in \mathbb{R}).$$

Since we also have $V_\lambda U(\lambda, t) = 0$ (cp. (3.4)), $x = \lambda$ is a multiple zero of $V_\lambda U(\cdot, t)$ for all $t \in \mathbb{R}$. Using Lemma 2.4, one shows easily that this is only possible if $V_\lambda U \equiv 0$. This in particular yields the desired conclusion:

$$V_\lambda u(\cdot, t_n) \to 0 \text{ in } C^1_{loc}(\mathbb{R}). \qquad \square$$

Take now the intervals $I_k := (-k, k)$, $k = 1, 2, \dots$, and let N_k be the number of zeros of $u_x(\cdot, t)$ in I_k for $t > T_1(I_k)$, where T_1 is as in Proposition 3.2. We distinguish the following mutually exclusive cases.

(C1) There is k_0 such that $N_k = 0$ for $k = k_0, k_0 + 1, \dots$.

(C2) There is k_0 such that $N_k = 1$ for $k = k_0, k_0 + 1, \dots$.

(C3) There is k_0 such that $N_k \geq 2$ for $k = k_0, k_0 + 1, \dots$.

According to Proposition 3.2, (C1) means that each bounded interval is free of critical points of $u(\cdot, t)$ for t large enough. In the case (C2), $u(\cdot, t)$ has exactly one critical point $\eta(t)$ such that $\eta(t)$ has a finite limit as $t \to \infty$; moreover, in any bounded interval, $u(\cdot, t)$ has no critical points different from $\eta_1(t)$ for t large enough. In the case (C3), there are more than one critical points of $u(\cdot, t)$ with finite limit as $t \to \infty$.

We give the proof of Theorem 1.1 in each of these cases separately.

3.2 Case (C1): no limit critical point

We consider case (C1) here. Clearly, (C1) implies that $u_x(\cdot, t)$ is of one sign in I_k for large t and this sign is independent of t. Without loss of generality, replacing $u(x, t)$ by $u(-x, t)$ if necessary, we assume that for all k one has

$$u_x(x,t) < 0 \quad (x \in (-k, k),\ t > T(I_k)). \tag{3.8}$$

In this situation, we have the following result concerning the ω-limit set $\omega(u)$:

Lemma 3.4. *Let ψ be any nonconstant periodic solution of* (1.6)*. Then*

$$\tau(\varphi) \cap \tau(\psi) = \emptyset \quad (\varphi \in \omega(u)).$$

Proof. We go by contradiction. Assume that there is $\varphi \in \omega(u)$ such that $\tau(\varphi) \cap \tau(\psi) \neq \emptyset$. This means that, possibly after replacing ψ by a translation, there exists $x_0 \in \mathbb{R}$ such that

$$\psi(x_0) = \varphi(x_0), \qquad \psi'(x_0) = \varphi'(x_0).$$

To simplify the notation, we will assume without loss of generality that $x_0 = 0$ (this can be achieved by a translation in the original equation, with no effect on the validity of the conclusion).

Let $U(x,t)$ be an entire solution as in (2.4). There exists a sequence $t_n \to \infty$ such that

$$u(\cdot, \cdot + t_n) \underset{n\to\infty}{\longrightarrow} U \text{ in } C^1_{loc}(\mathbb{R}^2).$$

Then the sequence $w_n := u(\cdot, \cdot + t_n) - \psi$ converges in $C^1_{loc}(\mathbb{R}^2)$ to the function $w(x,t) := U(x,t) - \psi(x)$. The function w solves a linear equation (2.5) and $w(\cdot, 0)$ admits a multiple zero at $x = 0$. Also, $w(\cdot, 0) = \varphi - \psi \not\equiv 0$, for ψ is nonconstant periodic, whereas for φ we have, as a direct consequence of (3.8), that $\varphi' \leq 0$. Applying Lemma 2.7, we obtain that there exist sequences $x_n \to 0$, $\delta_n \to 0$, such that $w_n(\cdot, \delta_n)$ has a multiple zero at $x = x_n$. Consequently, with $\tilde{t}_n := t_n + \delta_n$ we have $\tilde{t}_n \to \infty$, $x_n \to 0$, and

$$u(\cdot, \tilde{t}_n) - \psi \text{ has a multiple zero at } x = x_n. \tag{3.9}$$

We show that (3.9) contradicts (3.8).

Since ψ is periodic, there are $\rho_- < 0 < \rho_+$ such that $\psi(\rho_-) = \min \psi$, $\psi(\rho_+) = \max \psi$, and $x_n \in (\rho_-, \rho_+)$, for all n. Consider the function

$$z(t) := z_{(\rho_-, \rho_+)} \left(u(\cdot, t) - \psi \right).$$

By (3.8), there is $T_0 > 0$ such that for all $t > T_0$ we have $u_x(\cdot, t) < 0$ on $(\rho_- - 1, \rho_+ + 1)$. Thus, if $z(t) > 0$, then

$$u(\rho_-, t) > \psi(\rho_-) \text{ and } u(\rho_+, t) < \psi(\rho_+). \tag{3.10}$$

Therefore, Lemma 2.4 implies that

$$t \mapsto z(t) \text{ is nonincreasing on any subinterval of } \{t : t > T_0 \text{ and } z(t) > 0\}. \tag{3.11}$$

Assertion (3.9) implies that for some $T_1 > T_0$ one has $z(T_1) > 0$. We consider two complementary cases:

Case 1: $z(t) > 0$, for all $t > T_1$. In this case, (3.10) shows that Lemma 2.4 and Remark 2.5 apply to $v = u - \psi$. Therefore, for all large t, $u(\cdot, t) - \psi$ has only simple zeros in (ρ_-, ρ_+). This is a contradiction to (3.9).

Case 2: there exists $T_2 > T_1$ such that $z(T_2) = 0$. Pick large enough n_0 such that $\tilde{t}_{n_0} > T_2$ and set

$$T_3 := \sup\{t \in [T_2, \tilde{t}_{n_0}) : z(t) = 0\}.$$

From (3.9) we know that $T_3 < \tilde{t}_{n_0}$. The definition of T_3 and the monotonicity of $x \mapsto u(x, T_3)$ (cp. 3.8) implies that either $u(\rho_-, T_3) = \psi(\rho_-)$ or $u(\rho_+, T_3) = \psi(\rho_+)$. We only consider the first possibility, the other being similar. In addition to the relation $u(\rho_-, T_3) = \psi(\rho_-)$, we have, by (3.8) and the definition of ρ_-, $u_x(\rho_-, T_3) < 0 = \psi_x(\rho_-)$. The implicit function theorem therefore implies that there exists a continuous function $t \mapsto \eta(t)$ defined on a neighborhood of T_3 such that on a neighborhood of the point of $(\rho_-, T_3) \in \mathbb{R}^2$ one has $u(x, t) = \psi(x)$ if and only if $x = \eta(t)$. Using this, (3.8) and the continuity of u, we find $\varepsilon > 0$ such that $u(x, t) = \psi(x)$ holds with $x \in (\rho_- - \varepsilon, \rho_+]$, $t \in [T_3, T_3 + \varepsilon)$ only if $x = \eta(t)$. As a result, for $t > T_3$ close enough to T_3, we have $z(t) \leq 1$. The definition of T_3 implies that $z(t) > 0$ for all $t \in (T_3, t_{n_0}]$. From this obtain that, first, $z(t) = 1$ for all $t > T_3$, $t \approx T_3$, and, second, (3.11) applies to the interval $(T_3, t_{n_0}]$. Consequently, $z(t) = 1$ for all $t \in (T_3, t_{n_0}]$. Using this, (3.10), and (3.9) with $n = n_0$, we now obtain a contradiction to Lemma 2.4(iii) (take $t_0 = t_{n_0}$ in (2.6)). $\square$

Proof of theorem 1.1 in the case (C1). Assuming (3.8), we show that any $\varphi \in \omega(u)$ is either a constant steady state or a standing wave of (1.1).

By (3.8), $\varphi_x \leq 0$. Also, from Lemma 3.4 we know that, in the notation of Lemma 2.1, $\tau(\varphi) \subset \mathbb{R}^2 \setminus \mathcal{P}_0$.

If $\varphi_x \equiv 0$, then $\tau(\varphi)$ consists of the single point $(\varphi(0), 0)$. According to Lemma 2.1, this point is an equilibrium of (1.7) or is contained in $\tau(\psi)$, where ψ is a ground state solution of (1.6). In the later case, from Lemma 2.3 we obtain φ is a shift of ψ, which is impossible as $\varphi_x \equiv 0$. Thus, in this case, φ is a constant steady state.

Assume now that $\varphi_x \not\equiv 0$. We first show that $\varphi_x < 0$ on $\mathbb{R}$. Indeed, let $U(x, t)$ be the entire solution of (1.1) as in (2.4): $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. The latter implies that $U_x(\cdot, t) \leq 0$ for all t and the former gives $U_x(\cdot, 0) \not\equiv 0$. Therefore, applying the maximum principle to

U_x, we obtain $\varphi_x = U_x(\cdot, 0) < 0$, as desired. In particular, $\tau(\varphi)$ does not intersect the $u-$axis in the $u - v$ plane. This and Lemma 2.1(ii) imply that $\tau(\varphi) \subset \tau(\psi)$, where $\tau(\psi)$ is either a heteroclinic orbit of (1.7) or a homoclinic orbit of (1.7). The latter is impossible due to $\varphi_x < 0$. Thus $\tau(\psi)$ is a heteroclinic orbit of (1.7), meaning that ψ is a standing wave of (1.1). By Lemma 2.3, φ is a shift of ψ, hence it is a standing wave itself. □

3.3 Case (C2): a unique limit critical point

In the case (C2), there exists a C^1 function $t \mapsto \eta(t)$ defined on an interval (T_0, ∞) with with $\eta(t) \underset{t\to\infty}{\longrightarrow} \eta^\infty \in \mathbb{R}$ and with the following property. For each $k \in \{k_0, k_0+1, \dots\}$ there is $T(I_k)$ such that

$$\{(x,t) : u_x(x,t) = 0, x \in (-k,k),\ t > T(I_k)\} = \{(\eta(t),t) : t > T(I_k)\}. \tag{3.12}$$

Without loss of generality, using a shift if necessary, we will further assume that $\eta^\infty = 0$.

By Proposition 3.2, $x = \eta(t)$ is a simple zero of $u_x(\cdot,t)$, so $u(\cdot,t)$ has a strict local minimum or a strict local maximum at $\eta(t)$. We only consider the latter, the former is analogous. Thus, we henceforth assume that

$$u\left(\eta(t), t\right) = \max_{x\in(-k,k)} u(\cdot,t), \qquad t > T(I_k). \tag{3.13}$$

From (3.12), (3.13), and Lemma 3.3, we obtain that each $\varphi \in \omega(u)$ has the following properties:

$$\max_{\mathbb{R}} \varphi = \varphi(0), \quad \varphi_x(x) \le 0 \quad (x > 0), \quad \varphi(-x) = \varphi(x) \quad (x \in \mathbb{R}). \tag{3.14}$$

Moreover, for each $\varphi \in \omega(u)$

$$\text{either } \varphi_x \equiv 0 \text{ or } \varphi_x(x) < 0 \text{ for all } x > 0. \tag{3.15}$$

To prove this, let U be the entire solution of (1.1) with $U(\cdot,0) = \varphi$ and $U(\cdot,t) \subset \omega(u)$ for all $t \in \mathbb{R}$. Then, by (3.14), for all $t \in \mathbb{R}$ we have $U_x(0,t) = 0$ and $U_x(\cdot,t) \le 0$ on $[0,\infty)$. Since the function U_x satisfies a linear equation (2.5), the maximum principle implies that $\varphi_x = U_x(\cdot,0)$ is either identical to zero or strictly negative on $(0,\infty)$.

Our goal now is to prove the following result.

Proposition 3.5. *For some $\gamma \in \mathbb{R}$, one has $\{\varphi(0) : \varphi \in \omega(u)\} = \{\gamma\}$.*

Assuming that this true, we now complete the proof of Theorem 1.1 in the case (C2); we show that u is even convergent in this case. Then we give the proof of Proposition 3.5.

Proof of Theorem 1.1 in the case (C2). We prove that $\omega(u)$ consists of a single element, either a ground state or a constant steady state.

Given any $\varphi \in \omega(u)$. Let U be the entire solution of (1.1) with $U(\cdot, 0) = \varphi$, and $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. By Proposition 3.5, $U(0, t) = \gamma$ for all $t \in \mathbb{R}$. Therefore, $U_t(0, \cdot) \equiv 0$. Moreover, by (3.14), we have $U_x(0, t) = 0$ for all t, thus $U_{xt}(0, t) = 0$ for all t. This means that U_t has a multiple zero at $x = 0$ for all $t \in \mathbb{R}$. Since U_t satisfies a linear equation (2.5), Lemma 2.4 implies that $U_t \equiv 0$. This shows that $\varphi = U(\cdot, 0)$ is a steady state.

We have thus proved that every function $\varphi \in \omega(u)$ is a solution of the second order equation (1.6), satisfying $\varphi(0) = \gamma$ and $\varphi_x(0) = 0$. The uniqueness of this solution gives $\omega(u) = \{\varphi\}$, for some φ. Condition (3.14) together with the description of the solutions of (1.6) given in Subsection 2.1 imply that φ is either a ground state or a constant equilibrium. $\square$

For the proof of Proposition 3.5, we need several preliminary results. Denote

$$J := \{\varphi(0) : \varphi \in \omega(u)\}. \tag{3.16}$$

By compactness and connectedness of $\omega(u)$ in $L^\infty_{loc}(\mathbb{R})$, we have

$$J = [\gamma^-, \gamma^+] \text{ for some } \gamma^- \leq \gamma^+, \tag{3.17}$$

that is, J is a singleton or a compact interval. Proposition 3.5 says that $\gamma^- = \gamma^+$, so this is what we want to show at the end. First, we establish some properties of γ^-, γ^+. In the formulations of the lemmas below, we consider the open interval (γ^-, γ^+) with the understanding that it is empty if $\gamma^- = \gamma^+$.

Lemma 3.6. *The following assertions hold:*

(i) $f(s) > 0$ *for each* $s \in (\gamma^-, \gamma^+)$*;*

(ii) *if* $\gamma^- < \gamma^+$*, then* $\gamma^+ \in \omega(u)$ *and* $f(\gamma^+) = 0$.

(It is perhaps needless to say that in $\gamma^+ \in \omega(u)$, γ^+ refers to the constant function taking the value γ^+.)

Proof of Lemma 3.6. We prove (i) by contradiction. Assume there exists $s \in (\gamma^-, \gamma^+)$ with $f(s) \leq 0$ (in particular, $(\gamma^-, \gamma^+) \neq \emptyset$). Since $\eta(t) \to 0$, the definition of J implies that

$$\liminf_{t\to\infty} u(\eta(t), t) = \gamma^- < s, \qquad \limsup_{t\to\infty} u(\eta(t), t) = \gamma^+ > s. \tag{3.18}$$

Now, for all large t the function $u(\cdot, t)$ has a local maximum at $x = \eta(t)$. Therefore, equation (1.1) gives

$$(u(\eta(t), t))' = u_t(\eta(t), t) = u_{xx}(\eta(t), t) + f(u(\eta(t), t)) \leq f(u(\eta(t), t)).$$

This and the assumption $f(s) \leq 0$ imply, via an elementary comparison argument for the equation $\dot{\xi} = f(\xi)$, that if the relation $u(\eta(t), t) < 0$ is valid for some t, then it remains valid for all larger t. This contradiction to (3.18) proves statement (i).

For the proof of statement (ii), we assume that $\gamma^- < \gamma^+$. For a contradiction, we assume also that $\gamma^+ \notin \omega(u)$. Then, by relations (3.14), (3.15), and compactness of $\omega(u)$, there exists ε with $0 < \varepsilon < \gamma^+ - \gamma^-$ such that for all $\varphi \in \omega(u)$ one has $\varphi(\pm 1) < \gamma^+ - \varepsilon$. By statement (i), $f > 0$ on $(\gamma^+ - \varepsilon, \gamma^+)$. Therefore, we can choose $s \in (\gamma^+ - \varepsilon, \gamma^+)$ such that the solution ψ of (1.6) with $\psi(0) = s$, $\psi'(0) = 0$ is a nonstationary periodic solution of (1.6). (The existence of such s follows from Lemma 2.1, but one can also give more direct arguments, see for example [16, Lemma 3.2]). Let $\rho > 1$ be a period of ψ. Then $-\rho$ is also a period of ψ and we have

$$\psi(\pm\rho) = s, \qquad \varphi(\pm\rho) \leq \varphi(\pm 1) < s \quad (\varphi \in \omega(u)).$$

Hence, there is $T_1 > 0$ such that for all $t > T_1$ one has $u(\pm\rho, t) < s$. Therefore, by Lemma 2.4 and Remark 2.5, $z_{(-\rho,\rho)}(u(\cdot, t) - \psi)$ is finite for $t > T_1$ and for all sufficiently large t the function $u(\cdot, t) - \psi$ has only simple zeros in $(-\rho, \rho)$. On the other hand, the definition of $\gamma^\pm$ (cp. (3.16), (3.17)) yields $\varphi \in \omega(u)$ with $\varphi(0) = s = \psi(0)$. Since also $\varphi'(0) = 0$ (see (3.14)), $\varphi - \psi$ has a multiple zero at $x = 0$. Applying Lemma 2.7 as in the proof of Lemma 3.4, we conclude that there exist sequences $t_n \to \infty$, $x_n \to 0$ such that $u(\cdot, t_n) - \psi$ has a multiple zero at $x = x_n$, and we have a contradiction. This contradiction proves that $\gamma^+ \in \omega(u)$.

It remains to show that $f(\gamma^+) = 0$. If this is not true, then, by statement (i), $f(\gamma^+) > 0$. Let U be the entire solution of (1.1) with $U(\cdot, 0) \equiv \gamma^+$ and $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. Since U_x solves a linear equation (2.5), the identity $U_x(\cdot,0) \equiv 0$ implies that $U_x(\cdot,t) \equiv 0$ for all $t \in \mathbb{R}$. Thus $U = U(t)$ is a solution of $U_t = f(U)$ and $U'(0) = f(\gamma^+) > 0$. Thus $U(t) > \gamma^+$ for $t > 0$, which contradicts the definition of γ^+. The proof is now complete. $\square$

Lemma 3.7. *If $\lambda \in (\gamma^-, \gamma^+)$, then (the constant function) λ is not contained in $\omega(u)$.*

Proof. Assuming $\lambda \in \omega(u)$, let U be the entire solution of (1.1) with $U(\cdot,0) \equiv \lambda$ and $U(\cdot,t) \in \omega(u)$ for all $t \in \mathbb{R}$. Then, as at the end of the previous proof, $U = U(t)$ is a solution of $U_t = f(U)$, $U(0) = \lambda$. By Lemma 3.6(i), the range of the function U is an interval on which $f > 0$. We can choose $\tilde{\lambda}$ in this interval such that the solution ψ of (1.6) with $\psi(0) = \tilde{\lambda}$, $\psi'(0) = 0$ is a nonconstant periodic solution. This and symmetries of ψ (cp. (2.2)) imply that if $\rho > 0$ is the minimal period of ψ, then

$$\psi\left(\frac{\rho}{2}\right) = \min \psi, \qquad \psi(\rho) = \psi(2\rho) = \tilde{\lambda} = \max \psi.$$

We still have $\tilde{\lambda} \in \omega(u)$ as $\tilde{\lambda} = U\left(\tilde{t}\right)$ for some some $\tilde{t}$.

Now, as $\psi - U\left(\tilde{t}\right) = \psi - \tilde{\lambda}$ has a multiple zero at $x = \rho$ and $\psi - U \not\equiv 0$ (ψ is nonconstant), Lemma 2.7 yields sequences $t_n \to \infty$, $x_n \to \rho$ such that $u(\cdot,t_n) - \psi$ has a multiple zero at $x = x_n$ for all n. On the other hand, by (3.12), there exists $T_2 > 0$ such that for all $t > T_2$ one has $u_x(\cdot,t) < 0$ on $\left(\frac{\rho}{2}, 2\rho\right)$. One can now obtain a contradiction by considering $z_{\left(\frac{\rho}{2},2\rho\right)}(u(\cdot,t) - \psi)$ and using very similar arguments as in the proof of Lemma 3.4. We omit the details. $\square$

Lemma 3.8. *If $\lambda \in (\gamma^-, \gamma^+)$ and the solution ψ of (1.6) with $\psi(0) = \lambda$, $\psi'(0) = 0$ is periodic, then there is no $\varphi \in \omega(u)$ such that $\varphi \le \psi$ on $\mathbb{R}$.*

Proof. By Lemma 3.6, the periodic solution ψ is nonconstant. By Lemma 3.6(ii) and (2.2), ψ satisfies $\max \psi = \lambda$ and $\min \psi < \gamma^-$. Assume for a contradiction that there exists $\varphi \in \omega(u)$ such that $\varphi \le \psi$. Consider the following set

$$K := \{\xi \in \mathbb{R} : \text{there exists } \varphi \in \omega(u) \text{ such that } \varphi \le \psi(\cdot - \xi)\}. \tag{3.19}$$

By our assumption, K contains $\xi = 0$. By compactness of $\omega(u)$ in $L^\infty_{loc}(\mathbb{R})$, K is closed. We show that K is also open, thereby proving that actually $K = \mathbb{R}$.

Fix any $\xi \in K$ and take $\varphi \in \omega(u)$ as in (3.19). Let U be the entire solution of (1.1) with $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \omega(u)$ for all t. Since $\psi(\cdot - \xi)$ is a steady state of (1.1), the strong comparison argument gives

$$\tilde{\varphi} := U(\cdot, 1) < \psi(\cdot - \xi). \tag{3.20}$$

Since $\tilde{\varphi} \in \omega(u)$, relations (3.14) and the periodicity of ψ imply that relation (3.20) remains valid if ξ is replaced by $\tilde{\xi}$ with $\tilde{\xi} \approx \xi$. This shows the openness of K, hence $K = \mathbb{R}$.

Take now ξ such that $\psi(-\xi) = \min \psi$. For some $\varphi \in \omega(u)$ one has $\varphi \leq \psi(\cdot - \xi)$. In particular,

$$\varphi(0) \leq \psi(-\xi) = \min \psi < \gamma^-,$$

which is a contradiction (cp. (3.16), (3.17)). This contradiction completes the proof. □

As a direct corollary of Lemma 3.8(i), (3.15), and the compactness of $\omega(u)$ in $C^1_{loc}(\mathbb{R})$, we obtain the following result:

Corollary 3.9. *If $\lambda \in (\gamma^-, \gamma^+)$, then for all $\rho > 0$ there exist $\kappa > 0$ and $\varepsilon_1 \in (0, 1)$, depending on ρ, such that for any $\varphi \in \omega(u)$ with $|\varphi(0) - \lambda| \leq \kappa$ one has $\varphi' < -\varepsilon_1$ on $\left(\frac{\rho}{2}, \rho\right)$.*

Lemma 3.10. *If $\lambda \in (\gamma^-, \gamma^+)$ and the solution ψ of (1.6) with $\psi(0) = \lambda$, $\psi'(0) = 0$ is periodic, then there exist $T > 0$, $\varepsilon > 0$ with the following property. Denoting by $\rho > 0$ is the minimal period of ψ, we have*

$$z_{(-\rho,\rho)}(u(\cdot, t) - \psi) \leq 2 \tag{3.21}$$

whenever $t > T$ is such that $u(\pm\rho, t) \in (\psi(0) - \varepsilon, \psi)$.

Proof. As in the proof of Lemma 3.8, we have $\lambda = \psi(0) = \psi(\rho) = \max \psi$ and $\psi(\rho/2) = \min \psi$. Also recall that u_x is uniformly bounded for $t > 1$.

With $\kappa > 0$ and $\varepsilon_1 \in (0, 1)$ as in Corollary 3.9, we define the following positive quantities:

$$\varepsilon_2 = \min\left(\frac{\kappa}{2}, \frac{\rho\varepsilon_1}{8}\right), \qquad \delta = \frac{\varepsilon_2}{\|u_x\|_{L^\infty(\mathbb{R}\times(1,\infty))} + 1}.$$

Note that $\delta < \rho/8$. In particular, $\psi(\delta) < \psi(0)$. We will show that the conclusion of Lemma 3.10 is valid with

$$\varepsilon := \min\left(\psi(0) - \psi(\delta), \varepsilon_1, \varepsilon_2\right).$$

First, we claim that for any $\varphi \in \omega(u)$ with $\varphi(\rho) \in (\psi(0) - \varepsilon, \psi(0))$ one has $\varphi(0) > \psi(0) + \varepsilon_2$. Indeed, if not, then $\varphi(0) \le \psi(0) + \varepsilon_2 < \psi(0) + \kappa$. Since also $\varphi(\rho) > \psi(0) - \varepsilon > \psi(0) - \kappa$, using first Corollary 3.9 with the mean value theorem, and then (3.15), we obtain

$$\varphi(\rho) \le \varphi\left(\frac{\rho}{2}\right) - \varepsilon_1 \frac{\rho}{2} \le \varphi(0) - \varepsilon_1 \frac{\rho}{2} \le \psi(0) + \varepsilon_2 - \varepsilon_1 \frac{\rho}{2}.$$

However, since $\varepsilon_2 - \varepsilon_1 \rho/2 < -\varepsilon_2 \le \varepsilon$, we have a contradiction to the assumption $\varphi(\rho) \in (\psi(0) - \varepsilon, \psi(0))$. Thus, our claim is true.

In view of compactness of $\omega(u)$ and (2.3), the above claim implies that there exists $T > 1$ such that if $t > T$ and $u(\pm\rho, t) \in (\psi(0) - \varepsilon, \psi(0))$, then $u(0, t) > \psi(0) + \varepsilon_2$. Using our definition of δ and the mean value theorem, we infer from $u(0, t) > \psi(0) + \varepsilon_2$ that $u(\cdot, t) > \psi(0) \ge \psi$ on $(-\delta, \delta)$.

Next, we make T larger, if necessary, so as to guarantee that if $t > T$ we have $u_x(\cdot, t) > 0$ on $[-\rho, -\delta]$ and $u_x(\cdot, t) < 0$ on $[\delta, \rho]$ (cp. (3.12), (3.13)). Thus, if $t > T$ and $u(\pm\rho, t) \in (\psi(0) - \varepsilon, \psi(0))$, then for any $x \in [\delta, \rho/2]$ we have

$$u(x, t) > u(\rho, t) > \psi(0) - \varepsilon > \psi(\delta) \ge \psi(x),$$

where we have used the definition of ε and the monotonicity of ψ in $(0, \rho/2)$. Similarly one shows that $u(\cdot, t) > \psi$ on $[-\rho/2, \delta]$. Combining these estimates with the previous one, we conclude that $u(\cdot, t) - \psi > 0$ on $[-\rho/2, \rho/2]$ whenever $t > T$ and $u(\pm\rho, t) \in (\psi(0) - \varepsilon, \psi(0))$. Since the function $u(\cdot, t) - \psi$ is increasing on $\left(-\rho, -\frac{\rho}{2}\right)$ and decreasing on $\left(\frac{\rho}{2}, \rho\right)$, it can have at most one zero in each of this intervals. This implies the conclusion of the lemma (in fact, the conclusion holds with the equality sign in (3.21), but this is of no significance to us). $\square$

We can now complete the proof of Proposition 3.5

Proof of Proposition 3.5. The proof is by contradiction. Assume that $\gamma^- < \gamma^+$. As already noted in the proof of Lemma 3.6, we can then choose $\lambda \in (\gamma^-, \gamma^+)$ such that the solution ψ of (1.6) with $\psi(0) = \lambda$, $\psi'(0) = 0$ is nonconstant and periodic, and $\max \psi = \lambda < \gamma^+$. Let ρ be the minimal period of ψ and let ε, T be as in Lemma 3.10. Making $\varepsilon > 0$ smaller, with no effect on the conclusion of Lemma 3.10, we may assume that $\lambda - \varepsilon > \gamma^-$. Also, in view of (3.14), making T larger, if necessary, we may assume that

$$|u(-\rho, t) - u(\rho, t)| < \frac{\varepsilon}{2} \quad (t > T). \tag{3.22}$$

We next pick $s \in (\gamma^-, \lambda - \varepsilon)$. Then there is $\varphi \in \omega(u_0)$ with $\varphi(0) = s$. Lemma 3.6 rules out the possibility that $\varphi \leq \psi$ in $(-\rho, \rho)$. Therefore, using the evenness and periodicity of ψ in conjunction with (3.14), (3.15), one shows easily that $\psi - \varphi$ has at least 4 zeros in $(-\rho, \rho)$. Let now U be the entire solution of (1.1) with $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. For $t \approx 0$, we have

$$U(\pm\rho, t) \approx \varphi(\pm\rho) < \varphi(0) = s < \psi(0) - \varepsilon = \psi(\pm\rho) - \varepsilon.$$

Therefore, an application of Lemma 2.4 shows that arbitrarily close to 0 there is $t < 0$ such that $z_{(-\rho,\rho)}(\psi - U(\cdot, t)) \geq 4$ and all zeros of $\psi - U(\pm\rho, t)$ in $(-\rho, \rho)$ are simple. Replacing φ by $U(\cdot, t)$ for such t, we have thus found an element $\varphi \in \omega(u_0)$ such that

$$\varphi(\pm\rho) < \lambda - \varepsilon = \psi(0) - \varepsilon$$

and $\psi - \varphi$ has at least 4 simple zeros in $(-\rho, \rho)$.

Since γ^+, φ are elements of $\omega(u_0)$, we can approximate them arbitrarily closely in $C^1_{loc}(\mathbb{R})$ by $u(\cdot, T_1)$, $u(\cdot, T_2)$ with $T_2 > T_1 > T$. In particular, we can choose $T_2 > T_1 > T$ such that

$$u(\cdot, T_1) > \psi \text{ on } [-\rho, \rho] \tag{3.23}$$

and

$$z_{(-\rho,\rho)}(\psi - u(\cdot, T_2)) \geq 4, \quad u(\pm\rho, T_2) < \psi(0) - \varepsilon. \tag{3.24}$$

Denote

$$\tau := \inf\{s \in (T_1, T_2] : u(\pm\rho, t) < \psi(0) - \frac{\varepsilon}{2} \quad (s \leq t \leq T_2)\}.$$

By (3.24), τ is a well defined element of $[T_1, T_2)$. By (3.23), $\tau > T_1$. Therefore, at least one of the values $u(\pm\rho, \tau)$ is equal to $\psi(0) - \varepsilon/2$ and the relations $\tau > T_1 \geq T$ and (3.22) consequently give

$$u(\pm\rho, \tau) \in (\psi(0) - \varepsilon, \psi(0)).$$

It now follows from Lemma 3.10 that

$$z_{(-\rho,\rho)}(u(\cdot,\tau)-\psi) \le 2. \tag{3.25}$$

Since $u(\pm\rho,t) < \psi(0) = \psi(\rho)$ on $[\tau, T_2]$ (see the the definition of τ), the monotonicity of the zero number gives

$$z_{(-\rho,\rho)}(u(\cdot,T_2)-\psi) \le 2,$$

in contradiction to (3.24). This contradiction shows that $\gamma^- < \gamma^+$ is impossible, which completes the proof.

□

3.4 Case (C3): two or more limit critical points

In this last case, we assume that there exist two C^1 functions $\eta_1(t)$, $\eta_2(t)$ with $\eta_1(t) < \eta_2(t)$ and $\eta_i(t) \underset{t\to\infty}{\longrightarrow} \eta_i^\infty$, $i = 1, 2$, such that and for all t large enough one has

$$u_x(\eta_1(t),t) = u_x(\eta_2(t),t) = 0. \tag{3.26}$$

In view of Proposition 3.2, $\eta_1(t) < \eta_2(t)$ can be selected such that they are two successive critical points of $u(\cdot,t)$, one of them a local minimum point, the other one a local maximum point.

Lemma 3.11. *Set $\xi := \eta_2^\infty - \eta_1^\infty$. If $\xi > 0$, then each function $\varphi \in \omega(u)$ is $2\xi-$periodic.*

Proof. Lemma 3.3 implies that each function $\varphi \in \omega(u)$ is even about each of the two distinct points η_1^∞, η_2^∞. Therefore it is also even about the points $2\eta_1^\infty - \eta_2^\infty$, and $2\eta_2^\infty - \eta_1^\infty$. Repeating such reflections arguments one obtains the ξ-periodicity easily. □

Lemma 3.12. *Each $\varphi \in \omega(u_0)$ is a constant function.*

Proof. Suppose first that $\eta_1^\infty = \eta_2^\infty$. Since these are the limits of the critical points $\eta_1(t) < \eta_2(t)$, it follows that

$$\lim_{t\to\infty} u_{xx}(\eta_1^\infty,t) = \lim_{t\to\infty} u_x(\eta_1^\infty,t) = 0$$

Consequently, for each $\varphi \in \omega(u_0)$ we have $\varphi_{xx}(\eta_1^\infty) = \varphi_x(\eta_1^\infty) = 0$. Therefore, if U is the entire solution of (1.1) with $U(\cdot,0) = \varphi$ and $U(\cdot,t) \in \omega(u)$ for all

$t \in \mathbb{R}$, we have $U_x(\eta_1^\infty, t) = U_{xx}(\eta_1^\infty, t) = 0$ for all $t \in \mathbb{R}$. An application of Lemma 2.4 on a suitable interval, one shows easily that these relations can hold only if $U_x \equiv 0$. In particular, φ is constant.

Let now $\eta_1^\infty < \eta_2^\infty$. Suppose $\varphi \in \omega(u_0)$ and there is $\lambda \in \mathbb{R}$ with $\varphi_x(\lambda) \neq 0$. Then $x = \lambda$ is a simple zero of the function $V_\lambda(\varphi)(x) = \varphi(2\lambda - x) - \varphi(x)$. Lemma 3.11 implies that $V_\lambda(\varphi)$ is a periodic function, hence it has infinitely many simple zeros. Consequently, taking $t_n \to \infty$ such that $u(\cdot, t_n) \to \varphi$ in $C^1_{loc}(\mathbb{R})$, we have $z(V_\lambda u(\cdot, t_n)) \to \infty$. However, as noted in Subsection 3.1, the condition $u_0(-\infty) \neq u_0(\infty)$ implies that $z(V_\lambda u(\cdot, t))$ is bounded from above as $t \to \infty$. This contradiction completes the proof. $\square$

Proof of Theorem 1.1 in the case (C2). We show that $\omega(u_0) = \{\varphi\}$ for some constant φ. By Lemma 3.12—and compactness and connectedness—$\omega(u_0)$ is an interval $[a_1, b_1]$ of constants (which we identify here with the corresponding constant functions). Here $a_1 \leq b_1$ and we want to show that $a_1 = b_1$.

We go by contradiction. Assume $a_1 < b_1$. Then, clearly, there are $a < b$ such that $(a, b) \subset (a_1, b_1)$ and either $f \geq 0$ on (a, b) or $f \leq 0$ on (a, b). Assume the former, the latter is analogous.

For large t, one of points $\eta_1(t)$, $\eta_2(t)$, further denoted by $\eta(t)$, is a local minimum point of $u(\cdot, t)$. From the fact that $\omega(u_0)$ consists of constants, we infer that given any $k > 0$,

$$\sup_{x \in (-k,k)} |u(\eta(t), t) - u(x, t)| \underset{t \to \infty}{\longrightarrow} 0. \tag{3.27}$$

As $a, b \in \omega(u_0)$, (3.27) implies in particular that for some sequences $t_n \to \infty$, $t'_n \to \infty$ one has $u(\eta(t_n), t_n) \to b$, $u(\eta(t'_n), t'_n) \to a$. However, since $\eta(t)$ is a local minimum point and $f \geq 0$ on (a, b), equation (1.1) gives

$$(u(\eta(t), t))' = u_t(\eta(t), t) = u_{xx}(\eta(t), t) + f(u(\eta(t), t)) \geq 0,$$

whenever $u(\eta(t), t) \in (a, b)$. This implies that if n is so large that $u(\eta(t_n), t_n) > (a+b)/2$, then $u(\eta(t), t) > (a+b)/2$ for all $t > t_n$ and we have a contradiction.

This contradiction shows that $\omega(u_0)$ consists of a single constant λ. The invariance of $\omega(u_0)$ shows that this constant is a steady state of (1.1). The proof of Theorem 1.1 is now complete. $\square$

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