The influence of a line with fast diffusion on Fisher-KPP propagation: integral models

Antoine Pauthier

Institut de Mathématique de Toulouse

PhD supervised by Henri Berestycki (EHESS) and Jean-Michel Roquejoffre (IMT)
Supported by ERC ReaDi project
Presentation of the model(s)

Results
- An intermediate model

proof of the existence of $c_*$

Ongoing work
Model under study

The Field

The Road

fast diffusion: \( u_t - Du_{xx} = \) exchange terms

KPP reaction-diffusion
\[
v_t - d\Delta v = f(v) \pm \text{jump from the road}
\]
Mathematically

\[
\begin{aligned}
\partial_t u - D\partial_{xx} u &= -\bar{\mu} u + \int \nu(y) v(t, x, y) dy \\
\partial_t v - d\Delta v &= f(v) + \mu(y) u(t, x) - \nu(y) v(t, x, y)
\end{aligned}
\quad x \in \mathbb{R}, \ t > 0
\]

(1)

Assumptions :

- \( f(0) = f(1) = 0 \), \( f \) positive and concave on \( ]0, 1[ \) (KPP-type).
- \( \nu, \mu \geq 0 \), continuous, \( \nu(0) > 0 \).
- \( \exists M > 0, \mu(y) \leq Me^{-\frac{|y|}{M}} \) and \( \nu(y) \leq \frac{M}{(1+y^2)^{1+\frac{1}{M}}} \).
Motivation

Model introduced by H. Berestycki, J.-M. Roquejoffre, and L. Rossi.

The Road

fast diffusion: \( u_t - D u_{xx} = \) exchange terms

exchange terms

The Field

KPP reaction-diffusion

\( v_t - d \Delta v = f(v) \)

**Figure:** Road with fast diffusion
Mathematically

\[
\begin{align*}
\partial_t u - D\partial_{xx} u &= \nu v(x, 0, t) - \mu u & x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d\Delta v &= v(1 - v) & (x, y) \in \mathbb{R} \times \mathbb{R}_*, \ t > 0 \\
-d\partial_y v(x, 0, t) &= \mu u(x, t) - \nu v(x, 0, t) & x \in \mathbb{R}, \ t > 0.
\end{align*}
\]
Results for (2)

**Theorem**

There exists $c^* = c^*_*(\mu, d, D) > 0$ such that:
- for all $c > c^*$, $\lim_{t \to \infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$;
- for all $c < c^*$, $\lim_{t \to \infty} \inf_{|x| \leq ct} (u(x, t), v(x, y, t)) = (\nu/\mu, 1)$.

Moreover:
- if $D \leq 2d$, then $c^*_*(\mu, d, D) = c_{KPP} := 2\sqrt{df'(0)}$;
- if $D > 2d$, then $c^*_*(\mu, d, D) > c_{KPP}$ and $\lim_{D \to \infty} c^*_*(\mu, d, D)/\sqrt{D}$ exists and is positive.

**Question**

Do these results carry over to our model?
1. Presentation of the model(s)

2. Results
   - An intermediate model

3. proof of the existence of $c_*$

4. Ongoing work
Spreading

Theorem

(1) has a unique positive bounded stationary solution \((U_s(y), V_s(y))\)
\(x\)-invariant.

Theorem

There exists \(c^* = c^*(\mu, d, D) > 0\) s.t. :
- for all \(c > c^*\), \(\lim_{t \to \infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)\);
- for all \(c < c^*\), \(\lim_{t \to \infty} \inf_{|x| \leq ct} (u(x, t), v(x, y, t)) = (U_s, V_s)\).

Moreover, \(c^*\) satisfies :
- if \(D \leq 2d\), \(c^*(\mu, d, D) = c_{KPP} := 2\sqrt{df'(0)}\);
- if \(D > 2d\), \(c^*(\mu, d, D) > c_{KPP}\).

Remark

The threshold is still \(D = 2d\).
Singular limit $\mu, \nu \rightarrow \delta_0$

$\mu, \nu$ compactly supported. For all $\varepsilon > 0$, consider the exchange rates

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon} \nu\left(\frac{y}{\varepsilon}\right), \quad \mu_\varepsilon(y) = \frac{1}{\varepsilon} \mu\left(\frac{y}{\varepsilon}\right).$$

It gives a spreading speed $c^*_\varepsilon = c^*_\varepsilon(d, D, \bar{\mu})$. If $c^*_0$ is the spreading speed for (symmetrized) BRR-model, then

**Theorem**

$c^*_\varepsilon$ converges to $c^*_0$ with $\varepsilon \rightarrow 0$, locally uniformly in $d, D, \bar{\mu}$.
An intermediate model

- Integral exchange from the field to the road;
- Localized exchange from the road to the field.

\[
\begin{cases}
\partial_t u - D\partial_{xx} u = -\mu u + \int \nu(y) v(t, x, y) & x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d\Delta v = f(v) - \nu(y) v(t, x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, \ t > 0 \\
v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, \ t > 0 \\
-d \{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \mu u(t, x) & x \in \mathbb{R}, \ t > 0.
\end{cases}
\]

(3)

Similar results

Existence of an asymptotic spreading speed $c^*$ with the same properties.
1. Presentation of the model(s)

2. Results
   - An intermediate model

3. **proof of the existence of $c_*$**

4. Ongoing work
Main tool: construction of plane waves

Reminder: they serve as supersolution \((f(v) \leq f'(v)v)\).

Linearized system

\[
\begin{align*}
\partial_t u - D \partial_{xx} u &= -\bar{\mu} u + \int \nu(y) v(t, x, y) dy \quad x \in \mathbb{R}, \\
\partial_t v - d \Delta v &= f'(0)v + \mu(y) u(t, x) - \nu(y) v(t, x, y) \quad (x, y) \in \mathbb{R}^2,
\end{align*}
\]

Exponential solutions of the form:

\[
\begin{pmatrix} u(x, t) \\ v(x, y, t) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix},
\]

With positive \(\lambda, c, \phi \in H^1(\mathbb{R})\).
Equivalent system in $\lambda, \phi, c$

\[
\begin{align*}
-D\lambda^2 + \lambda c + \bar{\mu} &= \int \nu(y)\phi(y)\,dy \\
-d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) &= \mu(y).
\end{align*}
\]

- First equation $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \bar{\mu}$.
- Second equation: at most one $\phi(y; \lambda, c)$. Then set $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)\,dy$.

Goal

Find $\lambda, c$ such that the graphs of $\lambda \mapsto \Psi_1(\lambda)$ and $\lambda \mapsto \Psi_2(\lambda)$ intersect.
Graph of $\psi_1$

\[
\psi_1(\lambda) = \mu + \frac{c^2}{4D} \lambda^2
\]

**Figure:** graph of $\psi_1$
Study of $\psi_2$

System for $\lambda, c, \phi$

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}). \end{cases}$$

Existence and uniqueness for fixed $\lambda, c$ iff

$$\lambda c - d\lambda^2 - f'(0) > 0, \text{ soit } \lambda \in [\lambda_2^-(c), \lambda_2^+(c)]$$

with

$$\lambda_2^+(c) = \frac{c \pm \sqrt{c^2 - c_{KPP}^2}}{2d}, \quad c_{KPP} = 2\sqrt{df'(0)}.$$ 

Corollary

Travelling exponential supersolutions cannot exist for speed $c < c_{KPP}$. 

Antoine Pauthier (IMT)
ERC ReaDi meeting
March 25th 2014 16 / 32
Graph of $\lambda \mapsto \Psi_2(\lambda)$

**Figure:** Global vision of the graph of $\Psi_2$

**Proposition**

- **convexity and symmetry.**
- **vertical asymptote as** $\lambda \to \lambda_2^{\pm}$. 
When $c$ increases for $\Psi_1$

\[
\bar{\mu} + \frac{c^2}{4D}
\]

**Figure:** Movement of the parabola
When $c$ increases for $\Psi_2$

\[ \Psi_2(\lambda) \]

\[ \bar{\mu} \]

\[ \lambda_2^- \quad \frac{c}{2d} \quad \lambda_2^+ \]

**Figure:** Movement of the graph of $\Psi_2$
Case $D < 2d$

Existence of exponential travelling supersolutions at any speed $c > c_{KPP}$ (see Berestycki-Roquejoffre-Rossi)
Case $D > 2d : c < c_*$

\[
\psi_1(\lambda), \psi_2(\lambda)
\]

\[
\Gamma_1
\]

\[
\bar{\mu}
\]

\[
\frac{c}{D} \lambda_2^-
\]

**Figure:** $\text{Cas } D > 2d ; c_{KPP} < c < c_*$, no intersection

No solution
Case $D > 2d : c = c_*$

Exactly one solution
Case $D > 2d : c > c_*$

Two intersections, a range of exponential supersolutions (but two solutions of the linearized system)
Spread result, the limit $D \to \infty$

- Existence of an upper bound $c_\ast = c_\ast(d, D)$ for the spreading speed;
- if $D \leq 2d$, $c_\ast := c_{KPP} = 2\sqrt{df'(0)}$: no effect of the line;
- if $D > 2d$, $c_\ast > c_{KPP}$. The line enhances the spreading.

Subsolutions obtained by a perturbative method.

From geometrical considerations,

$$\sqrt{4\mu^2 + f'(0)^2} - 2\mu \leq \lim_{D \to \infty} \inf \frac{c^2_\ast(D)}{D} \leq \lim_{D \to \infty} \sup \frac{c^2_\ast(D)}{D} \leq f'(0).$$
The semi-limit model

Linearized system:

\[
\begin{aligned}
\partial_t u - D \partial_{xx} u &= v(x, 0, t) - \bar{\mu} u + \nu(y) v(t, x, y) & \quad x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d \Delta v &= f'(0) v - \nu(y) v(t, x, y) & \quad (x, y) \in \mathbb{R} \times \mathbb{R}^*, \ t > 0 \\
v(t, x, 0^+) &= v(t, x, 0^-), & \quad x \in \mathbb{R}, \ t > 0 \\
-d \{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} &= \bar{\mu} u(t, x) & \quad x \in \mathbb{R}, \ t > 0.
\end{aligned}
\]

Solutions of (6) of the form:

\[
\left( \begin{array}{c}
u(t, x) \\ v(t, x, y) \end{array} \right) = e^{-\lambda(x - ct)} \left( \begin{array}{c}1 \\ \phi(y) \end{array} \right)
\]
System in $\lambda, \phi$

\[
\begin{cases}
-D\lambda^2 + \lambda c + \overline{\mu} = \int \nu(y)\phi(y)dy \\
-d\phi_1''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_1(y) = 0 & y \geq 0. \\
-d\phi_2''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_2(y) = 0 & y \leq 0. \\
\phi_1(0) = \phi_2(0) \\
-\phi_1'(0) + \phi_2'(0) = \frac{\overline{\mu}}{d}.
\end{cases}
\]

Exactly the same method (up to the well-posedness of $\Psi_2$).
The singular limit

(Recall : \( \mu_\varepsilon(y) = \frac{1}{\varepsilon} \mu(y/\varepsilon), \nu_\varepsilon(y) = \frac{1}{\varepsilon} \nu(y/\varepsilon), \phi = \phi(y; \varepsilon, \lambda, c) \))

BRR model (2) : \( c_0^* \) given by the (first) intersection of algebraic curves in \((\alpha, \beta)\) plane :

\[
\begin{align*}
-D\alpha^2 + c\alpha &= \frac{\bar{\mu}}{1+2d\beta} - \bar{\mu} \\
-d\alpha^2 + c\alpha &= f'(0) + d\beta^2.
\end{align*}
\]

RP model (1) : \( c_\varepsilon^* \) given by the intersection of an algebraic and an implicit curve in \((\lambda, \int \nu_\varepsilon \phi)\) plane :

\[
\begin{align*}
-D\lambda^2 + \lambda c + \bar{\mu} &= \int \nu_\varepsilon(y)\phi(y)dy \\
-d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu_\varepsilon(y))\phi(y) &= \mu_\varepsilon(y).
\end{align*}
\]
**Figure:** BRR model $D > 2d$; $c < c_*$
Convergence of the curves with $\varepsilon \to 0$

The implicit curve goes to half of the circle with $\varepsilon \to 0$, the one corresponding to decreasing exponential in (2).

**Figure**: Case $D > 2d$; left: RP model; right: BRR model
1. Presentation of the model(s)

2. Results
   - An intermediate model

3. Proof of the existence of $c_*$

4. Ongoing work
The theorem we are investigating

Convergence of the solutions in the singular limit:

**Theorem**

- $c < c_0^*$, $\exists T_0$, $\exists \epsilon_0$ s.t. for all $\epsilon < \epsilon_0$, $t > T_0$,
  $$\inf_{|x| < ct} u(t, x) > \frac{1}{2\mu}.$$  

- $c > c_0^*$, $\forall \delta > 0$, $\exists T_\delta$, $\exists \epsilon_\delta$ s.t. for all $\epsilon < \epsilon_\delta$, $t > T_\delta$,
  $$\sup_{|x| > ct} u(t, x) < \delta.$$
Thank you for your attention!