The influence of a line with fast diffusion on Fisher-KPP propagation : integral models

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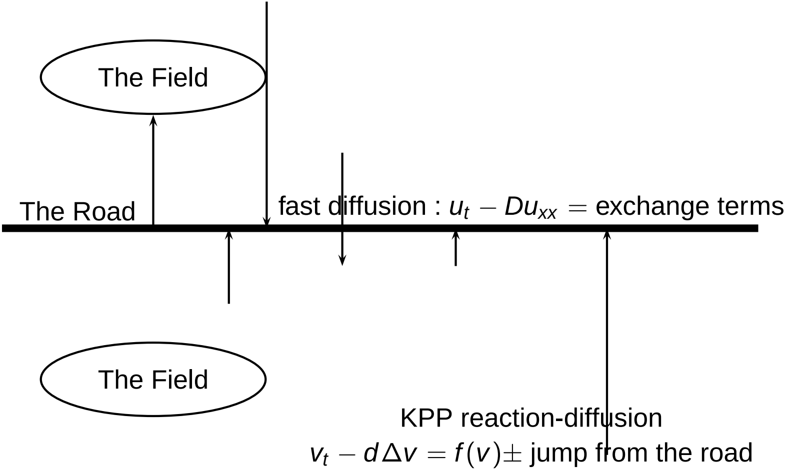
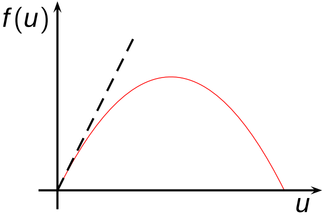
Model under study





Mathematically

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y) u(t,x) - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R}^2,\ t > 0 \end{cases} \qquad (1)$$

Assumptions :

- $f(0) = f(1) = 0,\ f$ positive and concave on $]0,1[$ (KPP-type).
- $\nu, \mu \geq 0$, continuous, $\nu(0) > 0$.
- $\exists M > 0,\ \mu(y) \leq Me^{-\frac{|y|}{M}}$ and $\nu(y) \leq \frac{M}{(1+y^2)^{1+\frac{1}{M}}}$.

Motivation

Model introduced by H. Berestycki, J.-M. Roquejoffre, and L. Rossi.

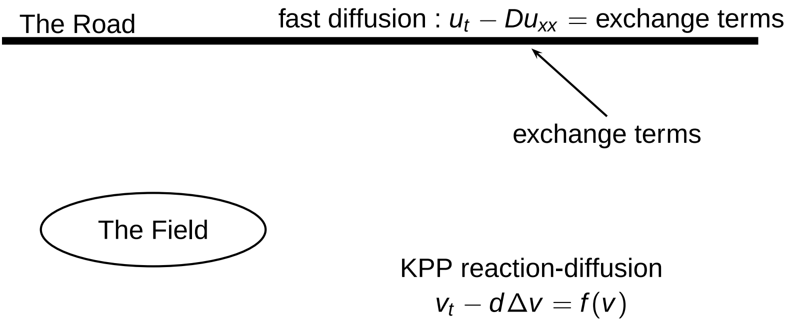


FIGURE: Road with fast diffusion

Mathematically

$$\begin{cases} \partial_t u - D\partial_{xx} u = \nu v(x,0,t) - \mu u & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = v(1-v) & (x,y) \in \mathbb{R} \times \mathbb{R}_-^*,\ t > 0 \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{2}$$

Results for (2)

Theorem

There exists $c_* = c_*(\mu, d, D) > 0$ *such that :*

- *for all* $c > c_*$, $\lim_{t\to\infty} \sup_{|x|\geq ct} (u(x,t), v(x,y,t)) = (0,0)$ *;*
- *for all* $c < c_*$, $\lim_{t\to\infty} \inf_{|x|\leq ct} (u(x,t), v(x,y,t)) = (\nu/\mu, 1)$.

Moreover :

- *if* $D \leq 2d$, *then* $c_*(\mu, d, D) = c_{KPP} := 2\sqrt{df'(0)}$ *;*
- *if* $D > 2d$, *then* $c_*(\mu, d, D) > c_{KPP}$ *and* $\lim_{D\to\infty} c_*(\mu, d, D)/\sqrt{D}$ *exists and is positive.*

Question

Do these results carry over to our model ?

Spreading

Theorem

(1) has a unique positive bounded stationary solution $(U_s(y), V_s(y))$ *x-invariant.*

Theorem

There exists $c_* = c_*(\mu, d, D) > 0$ *s.t. :*

- *for all* $c > c_*$, $\lim_{t\to\infty} \sup_{|x|\geq ct} (u(x,t), v(x,y,t)) = (0,0)$;
- *for all* $c < c_*$, $\lim_{t\to\infty} \inf_{|x|\leq ct} (u(x,t), v(x,y,t)) = (U_s, V_s)$.

Moreover, c^* *satisfies :*

- *if* $D \leq 2d$, $c_*(\mu, d, D) = c_{KPP} := 2\sqrt{df'(0)}$;
- *if* $D > 2d$, $c_*(\mu, d, D) > c_{KPP}$.

Remark

The threshold is still $D = 2d$.

Singular limit $\mu, \nu \to \delta_0$

μ, ν compactly supported. For all $\varepsilon > 0$, consider the exchange rates

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu(\frac{y}{\varepsilon}), \ \mu_\varepsilon(y) = \frac{1}{\varepsilon}\mu(\frac{y}{\varepsilon}).$$

It gives a spreading speed $c_\varepsilon^* = c_\varepsilon^*(d, D, \overline{\mu})$. If c_0^* is the spreading speed for (symmetrized) BRR-model, then

Theorem

c_ε^* converges to c_0^* with $\varepsilon \to 0$, locally uniformly in $d, D, \overline{\mu}$.

An intermediate model

- Integral exchange from the field to the road ;
- Localized exchange from the road to the field.

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R} \times \mathbb{R}^*,\ t > 0 \\ v(t,x,0^+) = v(t,x,0^-), & x \in \mathbb{R},\ t > 0 \\ -d\left\{\partial_y v(t,x,0^+) - \partial_y v(t,x,0^-)\right\} = \overline{\mu} u(t,x) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{3}$$

Similar results

Existence of an asymptotic spreading speed c^* with the same properties.

Main tool : construction of plane waves

Reminder : they serve as supersolution ($f(v) \leq f'(v)v$).

Linearized system

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R}, \\ \partial_t v - d\Delta v = f'(0)v + \mu(y)u(t,x) - \nu(y)v(t,x,y) & (x,y) \in \mathbb{R}^2, \end{cases} \tag{4}$$

Exponential solutions of the form :

$$\begin{pmatrix} u(x,t) \\ v(x,y,t) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}, \tag{5}$$

With positive λ, c, $\phi \in H^1(\mathbb{R})$.

Equivalent system in λ, ϕ, c

$$\begin{cases} -D\lambda^2 + \lambda c + \overline{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y). \end{cases}$$

- First equation $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \overline{\mu}$.
- Second equation : at most one $\phi(y; \lambda, c)$. Then set $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$.

Goal

Find λ, c such that the graphs of$\lambda \mapsto \Psi_1(\lambda)$ and $\lambda \mapsto \Psi_2(\lambda)$ intersect.

Graph of Ψ_1

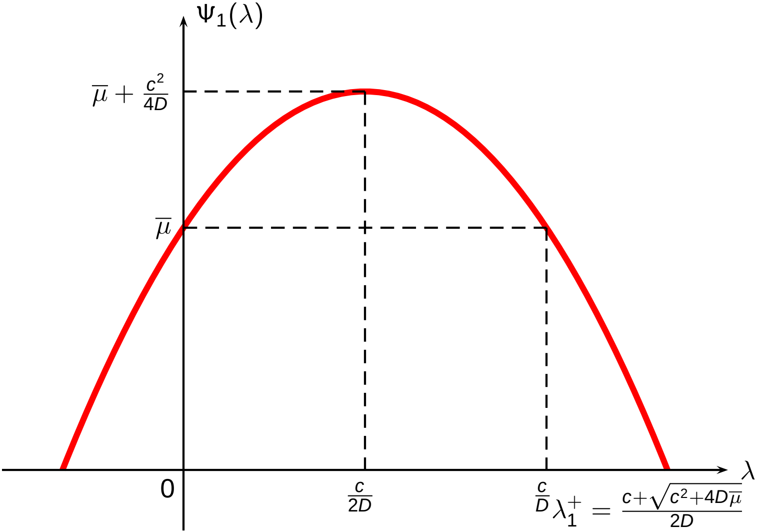


FIGURE: graph of Ψ_1

Study of Ψ_2

System for λ, c, ϕ

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}). \end{cases}$$

Existence and uniqueness for fixed λ, c iff

$$\lambda c - d\lambda^2 - f'(0) > 0, \text{ soit } \lambda \in]\lambda_2^-(c), \lambda_2^+(c)[$$

with

$$\lambda_2^{\mp}(c) = \frac{c \mp \sqrt{c^2 - c_{KPP}^2}}{2d}, \qquad c_{KPP} = 2\sqrt{df'(0)}.$$

Corollary

Travelling exponential supersolutions cannot exist for speed $c < c_{KPP}$.

Graph of $\lambda \mapsto \Psi_2(\lambda)$

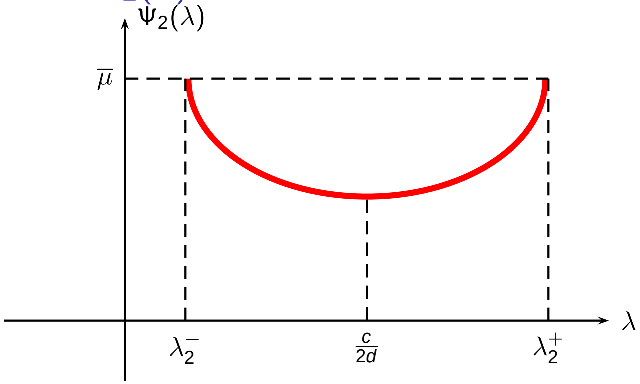


FIGURE: Gobal vision of the graph of Ψ_2

Proposition

- *convexity and symmetry.*
- *vertical asymptote as* $\lambda \to \lambda_2^{\pm}$.

When c increases for Ψ_1

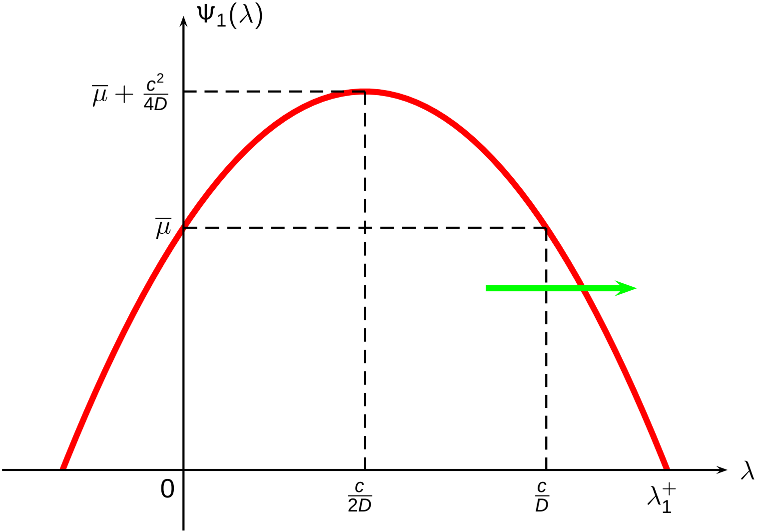


FIGURE: Movement of the parabola

When c increases for Ψ_2

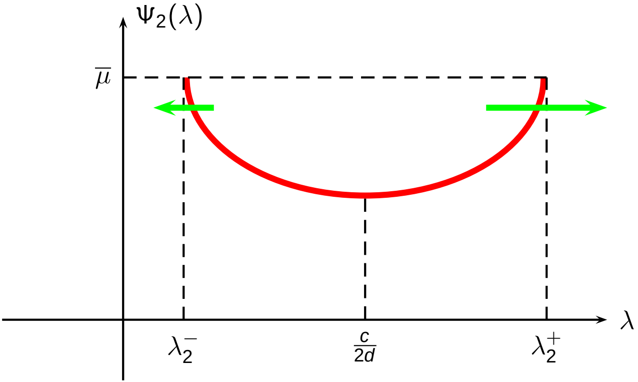


FIGURE: Movement of the graph of Ψ_2

Case $D < 2d$

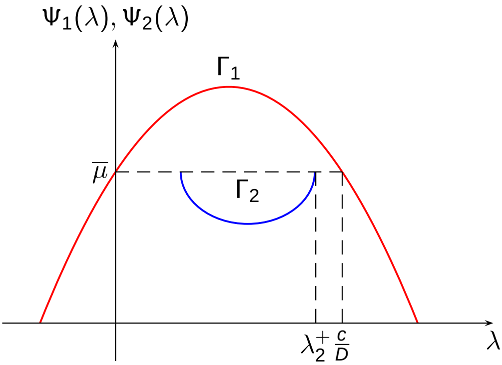


FIGURE: Cas $D < 2d$, $c > c_{KPP} = c_*$ not too large

Existence of exponential travelling supersolutions at any speed $c > c_{KPP}$ (see Berestycki-Roquejoffre-Rossi)

Case $D > 2d$: $c < c_*$

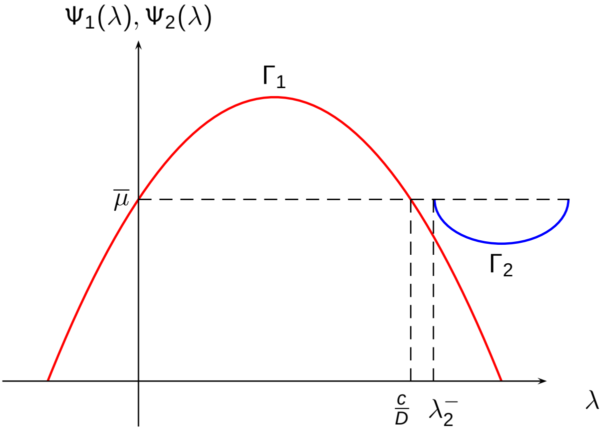


FIGURE: Cas $D > 2d$; $c_{KPP} < c < c_*$, no intersection

No solution

Case $D > 2d$: $c = c_*$

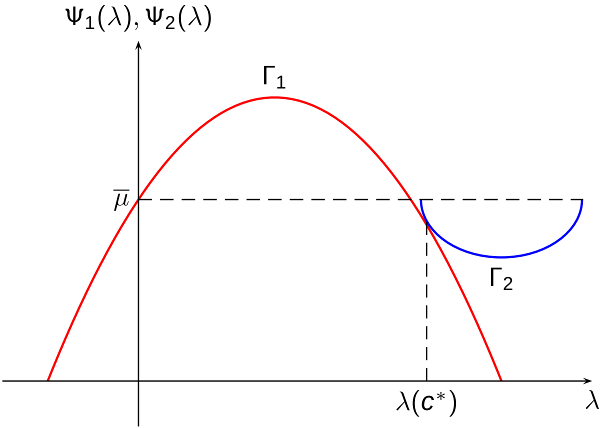


FIGURE: Case $D > 2d$; $c = c_*$, contact point

Exactly one solution

Case $D > 2d$: $c > c_*$

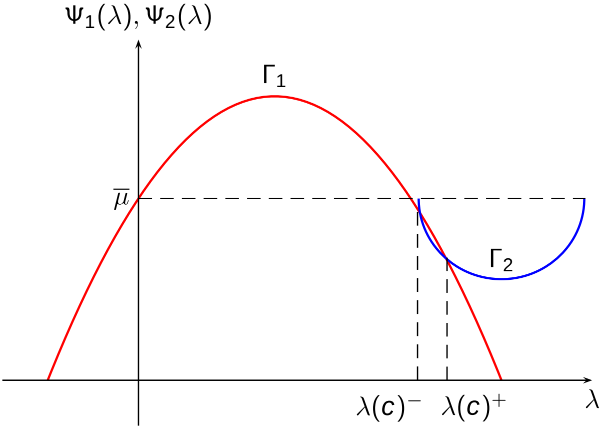


FIGURE: Case $D > 2d$; $c > c_*$, two intersections

Two intersections, a range of exponential supersolutions
(but two solutions of the linearized system)

Spreading result, the limit $D \to \infty$

- Existence of an upper bound $c_* = c_*(d, D)$ for the spreading speed ;
- if $D \le 2d$, $c_* := c_{KPP} = 2\sqrt{df'(0)}$: no effect of the line ;
- if $D > 2d$, $c_* > c_{KPP}$. The line enhances the spreading.
- Subsolutions obtained by a perturbative method.
- From geometrical considerations,

$$\sqrt{4\overline{\mu}^2 + f'(0)^2} - 2\overline{\mu} \le \liminf_{D\to\infty} \frac{c_*^2(D)}{D} \le \limsup_{D\to\infty} \frac{c_*^2(D)}{D} \le f'(0).$$

The semi-limit model

Linearized system :

$$\begin{cases} \partial_t u - D\partial_{xx} u = v(x,0,t) - \overline{\mu} u + \nu(y) v(t,x,y) & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f'(0) v - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R} \times \mathbb{R}^*,\ t > 0 \\ v(t,x,0^+) = v(t,x,0^-), & x \in \mathbb{R},\ t > 0 \\ -d\left\{\partial_y v(t,x,0^+) - \partial_y v(t,x,0^-)\right\} = \overline{\mu} u(t,x) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{6}$$

Solutions of (6) of the form :

$$\begin{pmatrix} u(t,x) \\ v(t,x,y) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}$$

System in λ, ϕ

$$\begin{cases} -D\lambda^2 + \lambda c + \overline{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi_1''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_1(y) = 0 & y \geq 0. \\ -d\phi_2''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_2(y) = 0 & y \leq 0. \\ \phi_1(0) = \phi_2(0) & i.e.\ \phi \text{ is continuous.} \\ -\phi_1'(0) + \phi_2'(0) = \frac{\overline{\mu}}{d}. \end{cases}$$

Exactly the same method (up to the well-posedness of Ψ_2).

The singular limit

(Recall : $\mu_\varepsilon(y) = \frac{1}{\varepsilon}\mu(\frac{y}{\varepsilon}),\ \nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu(\frac{y}{\varepsilon}),\ \phi = \phi(y;\varepsilon,\lambda,c)$)
BRR model (2) : c_0^* given by the (first) intersection of algebraic curves in (α,β) plane :

$$\begin{cases} -D\alpha^2 + c\alpha = & \frac{\overline{\mu}}{1+2d\beta} - \overline{\mu} \\ -d\alpha^2 + c\alpha = & f'(0) + d\beta^2. \end{cases}$$

RP model (1) : c_ε^* given by the intersection of an algebraic and an implicit curve in $(\lambda, \int \nu_\varepsilon \phi)$ plane :

$$\begin{cases} -D\lambda^2 + \lambda c + \overline{\mu} = \int \nu_\varepsilon(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu_\varepsilon(y))\phi(y) = \mu_\varepsilon(y). \end{cases}$$

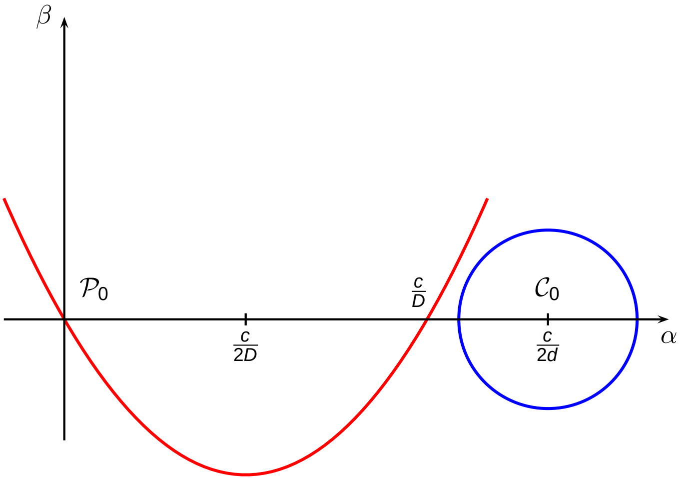


FIGURE: BRR model $D > 2d$; $c < c_*$

Convergence of the curves with $\varepsilon \to 0$

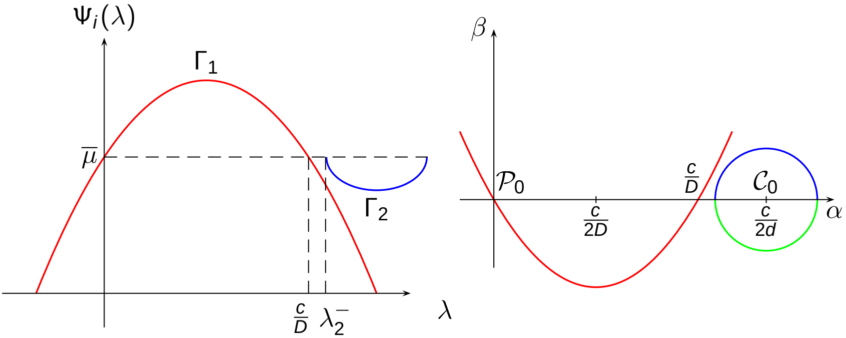


FIGURE: Case $D > 2d$; left : RP model ; right : BRR model

The implicit curve goes to **half** of the circle with $\varepsilon \to 0$, the one corresponding to decreasing exponential in (2).

The theorem we are investigating

Convergence of the solutions in the singular limit :

Theorem

- $c < c_0^*,\ \exists T_0,\ \exists \varepsilon_0$ *s.t. for all* $\varepsilon < \varepsilon_0, t > T_0,$

$$\inf_{|x|<ct} u(t,x) > \frac{1}{2\overline{\mu}}.$$

- $c > c_0^*,\ \forall \delta > 0,\ \exists T_\delta,\ \exists \varepsilon_\delta$ *s.t. for all* $\varepsilon < \varepsilon_\delta, t > T_\delta,$

$$\sup_{|x|>ct} u(t,x) < \delta.$$

Thank you for your attention !