1. (5 points) Using the Lagrange multiplier, find the maximum and minimum value of \( f(x, y) = 3x + y \) under the constraint \( x^2 + y^2 = 10 \).

Solution: Let \( g(x, y) = x^2 + y^2 - 10 \). First, we notice that \( \{(x, y) \in \mathbb{R}^2, g(x, y) = 0\} \) is a circle, hence a closed, bounded set in \( \mathbb{R}^2 \). The function \( f \) is continuous, it reaches its maximum and minimum on the circle, the problem has a solution.

A point of minimum or maximum for \( f \) under the constraint \( g = 0 \) has to satisfy for some real number \( \lambda \)
\[
\nabla f = \lambda \nabla g.
\]

It yields the following system:
\[
\begin{align*}
3 &= 2\lambda x \\
1 &= 2\lambda y \\
x^2 + y^2 &= 10.
\end{align*}
\]

Using the two first equations in the third gives
\[
\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10
\]
and then there are two possibilities \( \lambda = \frac{1}{2} \) and \( -\frac{1}{2} \), which gives two possible points of max and min for \( f \), \( P_1(3, 1) \) and \( P_2(-3, -1) \). Computation gives
\[
\min\{f(x, y), g = 0\} = f(P_2) = -10, \quad \max\{f(x, y), g = 0\} = f(P_1) = 10.
\]

2. (5 points) We consider the function \( f(x, y) = x^3 + y^3 - 3xy \) defined on \( \mathbb{R}^2 \).

(a) Find the critical points of \( f \).

Solution: The critical points of \( f \) are points \((x, y)\) that satisfy \( \nabla f(x, y) = 0 \). We have
\[
\nabla f(x, y) = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix}.
\]

Hence
\[
\nabla f(x, y) = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix} \iff \begin{cases} 3x^2 = 3y \\ 3y^2 = 3x \end{cases} \iff \begin{cases} x^2 = y \\ x^4 = x \end{cases} \iff \begin{cases} x^2 = y \\ x = 0 \text{ or } x = 1. \end{cases}
\]

The function \( f \) has two critical points, \( A(0, 0) \) and \( B(1, 1) \).

(b) Classify these critical points. Are they local minimum, local maximum, saddle points ?

Solution: We compute the following quantity at each critical point
\[
D(x, y) = \partial_{xx}f \partial_{yy}f - (\partial_{xy}f)^2.
\]
(Notice that the function is smooth, hence the cross derivatives are equal, Schwarz Theorem) We have
\[
\partial_{xx}f(x, y) = 6x, \quad \partial_{yy}f(x, y) = 6y, \quad \partial_{xy}f(x, y) = -3.
\]
Hence \( D(0, 0) = -9 < 0 \), the poit \( A(0, 0) \) is a saddle point; \( D(1, 1) = 27 > 0 \) and \( \partial_{xx}f(1, 1) = 6 > 0 \), the point \( B(1, 1) \) is a point of local minimum for \( f \).
Solution: No, the function $f$ can take any real value. Indeed, it is a continuous function and, for instance,

$$f(x, x) = 2x^3 - 3x^2 \xrightarrow{x \to \pm \infty} \pm \infty.$$