1. (5 points) Evaluate the triple integral \( I = \int_0^1 \int_y^{x+y} \int_0^x xydzdxdy. \)

**Solution:** The function \( f(x, y, z) = xy \) does not depend on \( z \). Hence we have

\[
I = \int_0^1 \int_y^{x+y} \int_0^x xydzdxdy = \int_0^1 \int_y^{x+y} (x^2 + xy)dxdy
\]

\[
= \int_0^1 \left[ \frac{x^3}{3} + \frac{yx^2}{2} \right]_y^{x+y} dy
\]

\[
= \int_0^1 \frac{23}{6}y^3dy = \frac{23}{30}.
\]

2. Let \( D \) be the lamina in the \( xy \)-plane bounded by the curves \( \{ y = x^2 \} \) and \( \{ y = 6 - \frac{x^2}{2} \} \). The density of the lamina is equal to the distance to the \( y \)-axis, id est given by \( \rho(x, y) = |x| \).

(a) (2 points) Sketch the domain \( D \) and give its representation in cartesian coordinates.

**Solution:** The domain is between the two parabolas \( \{ y = x^2 \} \) and \( \{ y = 6 - \frac{x^2}{2} \} \). They intersect at \((-2, 4)\) and \((2, 4)\).

Hence, a simple parametrisation of the domain is

\[
D = \{(x, y), \ -2 \leq x \leq 2, \ x^2 \leq y \leq 6 - \frac{x^2}{2}\}.
\]

Note that the domain is symmetric with respect to the axis \( \{x = 0\} \).

(b) (3 points) Using the symmetry of the domain, calculate the mass of the lamina \( M \).

**Solution:** The mass of the lamina is given by \( M = \iint_D \rho(x, y)dA \). We use the fact that \( x \mapsto \rho(x, y) = |x| \) is an even function and the symmetry of \( D \) with respect to the \( y \)-axis. It yields

\[
M = \int_{-2}^2 \int_{x^2}^{6-\frac{x^2}{2}} |x|dydx = 2 \int_0^2 \int_{x^2}^{6-\frac{x^2}{2}} xdydx
\]

\[
= 2 \int_0^2 x(6 - \frac{x^2}{2} - x^2)dx
\]

\[
= 2 \left( 3 \left[ x^2 \right]_0^0 - \frac{3}{8} \left[ x^4 \right]_0^0 \right) = 12.
\]

(c) (5 points) Using the symmetry of the domain, give the center of mass \((\bar{x}, \bar{y})\) of the lamina.
**Solution:** The center of mass \((\bar{x}, \bar{y})\) is given by

\[
\bar{x} = \frac{1}{M} \int \int_D x \rho(x, y) \, dA, \quad \bar{y} = \frac{1}{M} \int \int_D y \rho(x, y) \, dA.
\]

From symmetry of the domain and the eveness of \(\rho\) we immediately get \(\bar{x} = 0\). Again with the symmetry, we have

\[
\bar{y} = \frac{1}{M} \int_{-2}^{2} \int_{x^2}^{6-x^2} y |x| dy \, dx = \frac{2}{M} \int_{0}^{2} \int_{x^2}^{6-x^2} y x dy \, dx
\]

\[
= \frac{2}{M} \int_{0}^{2} \frac{x}{2} [y^2]_{x^2}^{6-x^2} dx = \frac{2}{M} \int_{0}^{2} \frac{x}{2} \left(6 - \frac{x^2}{2}\right)^2 - x^4 \right) dx
\]

\[
= \frac{2}{M} \int_{0}^{2} 18x - \frac{3}{8} x^5 - 3x^3 dx = \frac{2}{M} \left(18 \left[ x^2 \right]_0^2 - \frac{3}{48} \left[ x^6 \right]_0^2 - \frac{3}{4} \left[ x^4 \right]_0^2 \right)
\]

\[
= \frac{2}{M} \left(36 - 4 - 12\right) = \frac{40}{M} = \frac{10}{3} \approx 3.32.
\]