Math 2263 - Quiz 8
Line integrals

This is a 20 minutes quiz, two-sided. No calculators, computers, cellphones, notes, book allowed. Show all work. No credit will be given for answers without work.

1. (5 points) Compute the line integrals with respect to $x$

$$I = \int_C e^x \, dx$$

where $C$ is the arc of the curve $\{x = y^3\}$ from $(-1, -1)$ to $(1, 1)$.

**Solution:** A parametrization of the path $C$ could be given by

$$x(t) = t^3, \quad y(t) = t, \quad -1 \leq t \leq 1.$$ Hence we have

$$dx = x'(t)dt = 3t^2 dt,$$ and

$$I = \int_{-1}^{1} e^{t^3} 3t^2 dt = \left[ e^{t^3} \right]_{-1}^1 = e - \frac{1}{e}.$$  

2. We consider the following vector field

$$F: \begin{cases} \mathbb{D} \rightarrow \mathbb{R}^2 \\ (x, y) \rightarrow (2x \ln y - y \sin x)\hat{i} + (\cos x + \frac{x^2}{y})\hat{j} = P(x, y)\hat{i} + Q(x, y)\hat{j} \end{cases}$$

where $\mathbb{D} = \{(x, y), \ y > 0\}$.

(a) (3 points) Is it a conservative vector field? If yes, give a potential function $f(x, y)$ whose gradient is equal to $F$.

**Solution:** The vector field $F$ is clearly continuously differentiable on $\mathbb{D}$. To check if it is conservative, we compute $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$. We have:

$$\frac{\partial P}{\partial y} = \frac{2x}{y} - \sin x = \frac{\partial Q}{\partial x}$$

hence from Poincaré’s theorem the vector field is conservative. Thus, there exists potential functions $f$ such that $\nabla f = F$. Such a function must satisfy

$$\frac{\partial f}{\partial x} = P(x, y) = 2x \ln y - y \sin x, \quad \frac{\partial f}{\partial y} = Q(x, y) = \cos x + \frac{x^2}{y}.$$ (1)

We integrate the first relation in the $x-$variable. It gives

$$f(x, y) = x^2 \ln y + y \cos x + g(y).$$

for some function $g(y)$. We plug this in the second relation in (1), it gives $g'(y) = 0$ and we can choose any constant for $g$, for instance 0. Hence a potential function $f$ is given by

$$f(x, y) = x^2 \ln y + y \cos x.$$  

(b) (3 points) Compute $\int_C F \cdot dr$ where the curve $C$ is given by $r(t) = <\sqrt{t}, e^t>$, $0 \leq t \leq 4$.

**Solution:** The vector field is continuously differentiable on $\mathbb{D}$. We can apply the fundamental theorem for line integrals of vector field:

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = \int_0^4 \nabla f(r(t)).r'(t)dt = f(r(4)) - f(r(0)) = e^4 \cos 2 - 17.$$  

(c) (2 points) Is the set $\mathbb{D}$ open, connected, simply-connected? Justify.