Homework 2

MATH 8660 Fall 2019

Due by 11/25/2019

Q1. (circular law)

1. Let $G$ be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries. The eigenvalues (in random exchangeable order) of $G$ have joint density 

\[
\frac{1}{\pi^n \prod_{i=1}^n k!} \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{i<j} |z_i - z_j|^2,
\]

with respect to the Lebesgue measure on $\mathbb{C}^n$. Take this fact as granted.

(a) Show that the eigenvalues of $G$ form a determinantal point process on $(\mathbb{C}, \pi^{-1} e^{-|z|^2} dz)$ with kernel 

\[
K_n(z, w) = \sum_{k=0}^{n-1} \phi_k(z) \overline{\phi_k(w)}, \quad \text{where } \phi_k(z) := \frac{z^k}{\sqrt{k!}}.
\]

(b) Let $L_n$ and $\bar{L}_n$ be the ESD and the expected ESD of $n^{-1/2}G$ respectively. Write down the density of $\bar{L}_n$ with respect to Lebesgue measure on $\mathbb{C}$. Show that $\bar{L}_n \xrightarrow{d} \pi^{-1} 1_{\{|z|\leq 1\}} dz$. The limiting distribution is known as circular law.

(c) Prove that for each continuous bounded function $f$, 

\[
\mathbb{E}\left( \int f dL_n(z) - \int f d\bar{L}_n(z) \right)^4 = O(n^{-2}).
\]

(Note that to compute the LHS above, you need to know $m$-correlation functions of the eigenvalues of $G$ for $m \leq 4$.) Deduce that almost surely,

\[
L_n \xrightarrow{d} \pi^{-1} 1_{\{|z|\leq 1\}} dz.
\]

Figure 1: Plot of the real and imaginary parts (scaled by sqrt(1000)) of the eigenvalues of a 1000x1000 matrix with independent, standard normal entries (picture taken from wikipedia).
Q2. (linear statistics of CUE) Recall that the arguments of eigenvalues $\theta_1, \ldots, \theta_n$ of a $n \times n$ random Haar distributed unitary matrix (known as the Circular unitary Ensemble (CUE)) forms a determinantal point process with Kernel $K_n(s, t) = \frac{1}{2\pi} \sum_{k=0}^{n-1} e^{ikt(s-t)}$ with respect to the Lebesgue measure on $[-\pi, \pi]$.

Take $h : [-\pi, \pi] \to \mathbb{R}$ and define $N_n(h) = \sum_{k=1}^{n} h(\theta_k)$.

(a) Show that $\mathbb{E} N_n(h) = \frac{n}{2\pi} \int_{-\pi}^{\pi} h(t) \, dt$

and if $h$ has the Fourier expansion $h(t) = \sum_{k \in \mathbb{Z}} a_k e^{ikt}$, then

$$\text{Var}(N_n(h)) = \sum_{k \mid k \leq n} |k||a_k|^2 + n \sum_{k \mid k > n} |a_k|^2.$$

(b) Now take $h = 1_{[-\alpha, \alpha]}$ where $0 < \alpha < \pi$. Show that

$$\text{Var}(\chi_n([-\alpha, \alpha])) = \frac{1}{\pi^2} \log n + O(1).$$

(c) Conclude that

$$\frac{\chi_n([-\alpha, \alpha]) - \frac{n\alpha}{\pi}}{\pi^{-1/\sqrt{\log n}}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Q3. Using the steepest descent analysis, prove the following asymptotics of the Airy function

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{xz - z^3/3} \, dz \approx \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad \text{as } x \to \infty.$$