Hints/Solutions for Homework 4

MATH 8651 Fall 2015

Q1. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter $\lambda > 0$. Use Borel-Cantelli lemmas to show that

$$\limsup_n \frac{X_n}{\log n} = \frac{1}{\lambda}.$$  

[The p.d.f. of an exponential distribution with parameter $\lambda > 0$ is given by $f(x) = \lambda e^{-\lambda x} 1_{(x>0)}$.]

**Solution:** If $X \sim \text{exp}(\lambda)$, then $P(X > x) = \int_x^\infty \lambda e^{-\lambda t} dt = e^{-\lambda x}$ for each $x > 0$. For any $\epsilon > 0$,

$$\sum_{n=1}^\infty P\left(\frac{X_n}{\log n} > \frac{1+\epsilon}{\lambda}\right) = \sum_{n=1}^\infty e^{-(1+\epsilon)\log n} = \sum_{n=1}^\infty n^{-(1+\epsilon)} < \infty.$$  

By the first Borel-Cantelli lemma, $\frac{X_n}{\log n} > \frac{1+\epsilon}{\lambda}$ i.o. a.s. Hence, $\limsup_n \frac{X_n}{\log n} \leq \frac{1+\epsilon}{\lambda}$ a.s. for every $\epsilon > 0$. Hence $\limsup_n \frac{X_n}{\log n} \leq \frac{1}{\lambda}$ a.s. On the other hand,

$$\sum_{n=1}^\infty P\left(\frac{X_n}{\log n} > \frac{1-\epsilon}{\lambda}\right) = \sum_{n=1}^\infty e^{-(1-\epsilon)\log n} = \sum_{n=1}^\infty n^{-(1-\epsilon)} = \infty.$$  

Since the events $\frac{X_n}{\log n} > \frac{1-\epsilon}{\lambda}$ are independent, by the second Borel-Cantelli lemma, we have $\frac{X_n}{\log n} > \frac{1-\epsilon}{\lambda}$ i.o. a.s. Hence, $\limsup_n \frac{X_n}{\log n} \geq \frac{1-\epsilon}{\lambda}$ a.s. for every $\epsilon > 0$. Hence $\limsup_n \frac{X_n}{\log n} \geq \frac{1}{\lambda}$ a.s. Combining the above two inequalities, we obtain

$$\limsup_n \frac{X_n}{\log n} = \frac{1}{\lambda} \text{ a.s.}$$

Q2. Prove the following variant of the first Borel-Cantelli lemma.

If $P(A_n) \to 0$ and $\sum_{n=1}^\infty P(A_n \cap A_{n+1}) < \infty$ then $P(A_n \text{ i.o.}) = 0$.

**Solution:** Let $X_n = 1_{A_n}$. Note that if $X_n = 1$ occurs infinitely often, then either there are infinitely many switches from 1 to 0 in the sequence $(X_n)_{n \geq 1}$ or $X_n = 1$ for all but finitely many $n$. Writing in this terms of the events $A_n$, we obtain

$$\{A_n \text{ i.o.}\} = \{A_n \cap A_{n+1}^c \text{ i.o.}\} \cup \liminf A_n.$$  

Thus, we have $P(A_n \text{ i.o.}) \leq P(A_n \cap A_{n+1}^c \text{ i.o.}) + P(\liminf A_n)$. By the first Borel-Cantelli lemma, $P(A_n \cap A_{n+1}^c \text{ i.o.}) = 0$ On the other hand, $P(\liminf A_n) \leq \liminf P(A_n) = 0$.

Q3. Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_1| < \infty$. Let $M_n = \max(X_1, \ldots, X_n)$. Prove that $n^{-1}M_n \to 0$ a.s.

**Solution:** For $\epsilon > 0$,

$$\sum_n P(n^{-1}|X_n| > \epsilon) = \sum_n P(|\epsilon^{-1}X_n| > n) \leq \int_0^\infty P(|\epsilon^{-1}X_1| > t) dt = E|\epsilon^{-1}X_1| < \infty,$$  

where the last equality follows from an application of Fubini’s theorem. Hence, by Borel-Cantelli, we have

$$\limsup n^{-1}|X_n| \leq \epsilon \text{ a.s.}$$

This implies that

$$\limsup n^{-1} \max_{1 \leq i \leq n} |X_i| \leq \epsilon \text{ a.s.}$$

This is true for each $\epsilon > 0$. Thus, $\limsup n^{-1} \max_{1 \leq i \leq n} |X_i| = 0$ a.s. and therefore $n^{-1}M_n = 0$ a.s.
Q4. Suppose an unbiased coin is flipped independently infinitely many times. Let $T_n$ be the minimum number of flips to get $n$ heads. Show that $T_n/n \to c$ a.s. and compute $c$.

**Solution:** We can write $T_n = X_1 + X_2 + \ldots + X_n$, where $X_i$ is the number of new tosses required after the $(i-1)$th head to get the $i$th head. Clearly, each $X_i$ is i.i.d. with Geo($\frac{1}{2}$) distribution. Thus, by SLLN, $n^{-1}T_n \xrightarrow{a.s.} \mathbb{E}X_1 = 2$.

If you don’t want to assume that $X_i$ is i.i.d. with Geo($\frac{1}{2}$) distribution (which, actually, requires a proof), you can follow the argument of example 2.4.1 of Durrett (renewal theory example). Let $S_k$ be the number of heads in first $k$ tosses. By SLLN, $\frac{S_k}{k} \to \frac{1}{2}$ a.s. Note that $T_n \geq n$, therefore $T_n \uparrow \infty$ pointwise. So,

$$\frac{S_{T_n}}{T_n} \to \frac{1}{2} \quad \text{a.s.}$$

Note that $S_{T_n} = n$, so we can write

$$\frac{T_n}{n} = \frac{T_n}{S_{T_n}} \to 2 \quad \text{a.s.}$$

Q5. Durrett 2.3.6

**Solution:** Suppose, if possible, $\liminf_n \mathbb{E}X_n < \mathbb{E}X$. So, there exists a subsequence $\{n_k\}$ such that $\mathbb{E}[X_{n_k}] \leq \mathbb{E}[X] - \epsilon$ for each $k$ for some $\epsilon > 0$. Find a further subsequence $\{n_{k_l}\}$ such that $X_{n_{k_l}} \xrightarrow{a.s.} X$. By Fatou’s lemma, we have that $\liminf_k \mathbb{E}[X_{n_{k_l}}] \geq \mathbb{E}[X]$, a contradiction.

Q6. Durrett 2.4.3

**Solution:** Let $W_{n+1} = |X_{n+1}|/|X_n|$. Then, $W_1, W_2, \ldots$ are i.i.d. distributed according to the distance of a random point on the unit disc from the origin. We have $|X_n| = \prod_{i=1}^{n} W_i$. By SLLN, $n^{-1} \log |X_n| = n^{-1} \sum_{i=1}^{n} \log W_i \to c = \mathbb{E}[\log W_1]$ a.s. (provided $\mathbb{E}[|\log W_1|] < \infty$). Let us now compute $\mathbb{E}[\log W_1]$. Let $(U, V)$ be a point chosen uniformly at random from the unit disc, i.e., $(U, V)$ has a bivariate density

$$f(u, v) = \pi^{-1}1_{\{u^2 + v^2 \leq 1\}}.$$ 

Since $W_1 \overset{d}{=} \sqrt{U^2 + V^2}$, 

$$\mathbb{E}[\log W_1] = \mathbb{E}\frac{1}{2} \log(U^2 + V^2) = \int \int \frac{1}{2} \log(u^2 + v^2) f(u, v) dudv$$

$$= \pi^{-1} \int \int_{\{u^2 + v^2 \leq 1\}} \frac{1}{2} \log(u^2 + v^2) dudv$$

$$= \pi^{-1} \int_{0}^{2\pi} \int_{0}^{1} \log r \cdot r drd\theta$$

$$= 2 \int_{0}^{1} \log r \cdot rdr = -\frac{1}{2}.$$

Since $0 \leq W_1 \leq 1$, $|\log W_1| = -\log W_1$ and thus by above calculation, $\mathbb{E}[\log W_1] = \frac{1}{2} < \infty$, proving the validity of SLLN.

Q7. Durrett 2.4.4

**Solution:** (i) The SLLN guarantees that $n^{-1} \log W_n \to c(p) := \mathbb{E}\log(ap + (1-p)V_1)$ a.s. (to show the finiteness of $\mathbb{E}[\log(ap + (1-p)V_1)]$, use $|\log(ap + (1-p)V_1)| \leq \max\{ap + (1-p)V_1, \frac{1}{(1-p)V_1}\}$.)

(ii) By Theorem A.5.1 of Durrett, we can exchange differentiation and expectation. We obtain

$$c'(p) = \mathbb{E}\left(\frac{a - V_1}{ap + (1-p)V_1}\right) \quad \text{and} \quad c''(p) = -\mathbb{E}\left(\frac{a - V_1}{ap + (1-p)V_1}^2\right)$$

Since $c'' \leq 0$, $c$ is concave.

(iii) In order to have the maximum of $c$ inside $(0, 1)$, we need to have $c'(0) > 0$ and $c'(1) < 0$, that is, $\mathbb{E}[V_1^{-1}] > a^{-1}$ and $\mathbb{E}[V_1] > a$. 

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(iv) In this case, $EV_{1^{-1}} = \frac{5}{8}$ and $EV_1 = \frac{5}{2}$. Thus, when $a \leq \frac{5}{2}$, the maximum of $c$ occurs at $p = 0$ and when $a \geq \frac{5}{2}$, the maximum of $c$ occurs at $p = 1$. In between, the maximum occurs for $p \in (0, 1)$ such that $c'(p) = 0$, that is,

$$\frac{a-1}{2 ap+(1-p)} + \frac{a-4}{2 ap+4(1-p)} = 0 \iff p = \frac{5a-8}{2(a-1)(4-a)}.$$ 

Q8. Durrett 2.5.9

[Hint for Durrett 2.5.9: For $m < j \leq n$ if $|S_{m,j}| > 2a$ and $|S_{j,n}| \leq a$ then $|S_{m,n}| > a$.]

**Solution:** Fix $m < n$. Let $T := \min\{j \geq m + 1 : |S_{m,j}| > 2a\}$. Using the hint, it follows that

$$\{ |S_{m,n}| > a \} \supset \bigcup_{m < j \leq n} \{ T = j, |S_{j,n}| \leq a \}.$$ 

Note that the events $\{ T = j, |S_{j,n}| \leq a \}$ are disjoint for $m < j \leq n$ and $\{ T = j \} \in \sigma(X_{m+1}, \ldots, X_n)$ is independent of $\{ |S_{j,n}| \leq a \} \in \sigma(X_{j+1}, \ldots, X_n)$. Therefore,

$$P(|S_{m,n}| > a) \geq \sum_{j=m+1}^{n} P(T = j, |S_{j,n}| \leq a) = \sum_{j=m+1}^{n} P(T = j)P(|S_{j,n}| \leq a)$$

$$\geq \min_{m < k \leq n} P(|S_{k,n}| \leq a) \sum_{j=m+1}^{n} P(T = j) = \min_{m < k \leq n} P(|S_{k,n}| \leq a)P(m < T \leq n)$$

$$= \min_{m < k \leq n} P(|S_{k,n}| \leq a)P( \max_{m < j \leq n} |S_{m,j}| > 2a).$$

Q9. Durrett 2.5.10

**Solution:** Let $S_n$ converges in probability to $S_\infty$. Fix $m \in \mathbb{N}$ and $a > 0$. From the previous exercise, for any $n > m$,

$$P( \max_{m < k \leq n} |S_j - S_m| > 2a) \leq P(|S_n - S_m| > a) \leq \min_{m < k \leq n} P(|S_n - S_k| \leq a).$$

Now $P(|S_n - S_m| > a) \leq P(|S_n - S_\infty| > a/2) + P(|S_m - S_\infty| > a/2)$ and taking $n \to \infty$ (and keeping $m$ fixed) we have $\limsup_n P(|S_n - S_m| > a) \leq P(|S_m - S_\infty| > a/2)$. On the other hand,

$$\min_{m < k \leq n} P(|S_n - S_k| \leq a) = 1 - \max_{m < k \leq n} P(|S_n - S_k| > a)$$

$$\geq 1 - \max_{m < k \leq n} (P(|S_k - S_\infty| > a/2) + P(|S_n - S_\infty| > a/2))$$

$$\geq 1 - 2 \max_{m < k \leq n} P(|S_k - S_\infty| > a/2).$$

Now, by taking $n \to \infty$ in (1), we obtain

$$P( \sup_{j \geq m} |S_j - S_m| > 2a) \leq \frac{P(|S_m - S_\infty| > a/2)}{1 - 2 \sup_{k > m} P(|S_k - S_\infty| > a/2)}.$$ 

Now taking $m \to \infty$, we have $P(|S_m - S_\infty| > a/2) \to 0$ and $\sup_{k > m} P(|S_k - S_\infty| > a/2) = 0$. Hence,

$$\limsup_m P( \sup_{j \geq m} |S_j - S_m| > 2a) = 0.$$ 

Finally, using

$$P( \sup_{j, \ell \geq m} |S_j - S_\ell| > 4a) \leq 2P( \sup_{j \geq m} |S_j - S_m| > 2a),$$

we see that $T_m := \sup_{j, \ell \geq m} |S_j - S_\ell| \overset{P}{\to} 0$ as $m \to \infty$. Since $T_m$ is decreasing, we have $T_m \overset{a.s.}{\to} 0$ and hence $S_n$ is almost surely Cauchy and hence converges almost surely.