Q1. Let $0 < a_1 < a_2 < \cdots$ be fixed real numbers and let $(X_k)_{k \geq 1}$ be a sequence of i.i.d. random variables with zero mean and unit variance. Let $T_n = \sum_{i=1}^{n} a_i X_i$. Find a simple sufficient (deterministic) condition on the sequence $(a_k)_{k \geq 1}$ to ensure that

$$\frac{T_n}{\sqrt{\text{Var}(T_n)}} \xrightarrow{d} N(0,1).$$

Use your sufficient condition to verify the asymptotic normality of $T_n$ when $a_k = k^\alpha$ where $\alpha > 0$ is a fixed real number.

Q2. Suppose that $X_n \xrightarrow{d} X$. Give a counterexample to the following statement: If $B$ is a Borel subset of $\mathbb{R}$, and $P(X_n \in B) = 1$ for all $n$, then $P(X \in B) = 1$.

Show that, however, if $B$ is closed subset of $\mathbb{R}$, then the above statement is true.

Q3. If $X_n \xrightarrow{d} X$ and $X_n \xrightarrow{d} Y$, show that $X$ and $Y$ have the same distribution.

[Comment: We are not claiming here that $X = Y$ which is obviously false (why?).]

Q5- Q9. Durrett 3.2.3, 3.2.11, 3.2.12 (converse part), 3.2.14, 3.4.2 (i), 3.4.4, 3.4.5