Q1. Let $\mu_1, \mu_2, \mu_3$ be three probability measures on $(\Omega, \mathcal{F})$ such that

$$\mu_1 \ll \mu_2 \ll \mu_3.$$ 

Show that

$$\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2} \times \frac{d\mu_2}{d\mu_3} \text{ a.s.}$$

Solution: Note that $\frac{d\mu_1}{d\mu_2} \times \frac{d\mu_2}{d\mu_3}$ is $\mathcal{F}$-measurable. Since $\mu_2 \ll \mu_3$,

$$\mu_2(A) = \int_A \frac{d\mu_2}{d\mu_3} d\mu_3,$$

for any $A \in \mathcal{F}$. Consequently, by Durrett exercise 1.6.8 (see Homework 2, Q9), for any measurable function $g \geq 0$,

$$\int g d\mu_2 = \int g \frac{d\mu_2}{d\mu_3} d\mu_3,$$

(1)

Now since $\mu_1 \ll \mu_2$, for $A \in \mathcal{F}$,

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2} d\mu_2 = \int \frac{d\mu_1}{d\mu_2} \times \frac{d\mu_2}{d\mu_3} d\mu_3,$$

where we take $g = 1_A \frac{d\mu_1}{d\mu_2}$ in (1) to obtain the last equality above. The claim now follows from the (almost sure) uniqueness of Radon-Nikodym derivatives.

Q2. Let $X$ and $Y$ be independent, identically distributed integrable random variables. Show that

$$\mathbb{E}[X|X+Y] = \frac{X+Y}{2}.$$ 

Solution: We claim that $\mathbb{E}[X|X+Y] = \mathbb{E}[Y|X+Y]$ almost surely. To prove the claim, we need to show that

$$\int_{X+Y \in B} \mathbb{E}[X|X+Y] dP = \int_{X+Y \in B} X dP = \int_{X+Y \in B} Y dP = \int_{X+Y \in B} \mathbb{E}[Y|X+Y] dP \text{ for any } B \in \mathcal{B}_\mathbb{R}.$$ 

Note that since $X$ and $Y$ are i.i.d., $(X,Y) \overset{d}{=} (Y,X)$, implying that $\mathbb{E}f(X,Y) = \mathbb{E}f(Y,X)$ for any measurable function $f: \mathbb{R}^2 \to \mathbb{R}$ such that $|f(X,Y)| < \infty$.

Applying the above identity to the function $f(u,v) = u 1_{u+v \in B}$ and noting that $f(X,Y)$ is integrable since $|f(X,Y)| \leq |X|$, we obtain that

$$\int_{X+Y \in B} X dP = \mathbb{E}f(X,Y) = \mathbb{E}f(Y,X) = \int_{X+Y \in B} Y dP.$$ 

Hence, the claim follows. Now, almost surely,

$$\mathbb{E}[X|X+Y] = \mathbb{E}[Y|X+Y] = \frac{1}{2} (\mathbb{E}[X|X+Y] + \mathbb{E}[Y|X+Y]) = \frac{1}{2} \mathbb{E}[X+Y|X+Y] = \frac{1}{2} (X+Y).$$
Q3. Let $X$ and $Y$ be integrable random variables and suppose that

$$\mathbb{E}[X|Y] = Y \text{ and } \mathbb{E}[Y|X] = X \text{ a.s.}$$

Show that $X = Y$ a.s.

[Comment: When $X, Y \in L^2$, the above result is an easy consequence of Durrett exercise 5.1.11 by taking $\mathcal{G} = \sigma(X)$. But in the above problem, we only assume $X, Y \in L^1$. Hint: Work with quantities like $\mathbb{E}[(X - Y)1_{(X > a, Y \leq a)}] + \mathbb{E}[(X - Y)1_{(X \leq a, Y > a)}]$ for $a \in \mathbb{R}$.]

Solution: Since $\{Y \leq a\} \in \sigma(Y)$ and $\mathbb{E}[X|Y] = Y$ a.s.,

$$\mathbb{E}[Y1_{Y \leq a}] = \mathbb{E}[\mathbb{E}[Y|X]1_{Y \leq a}] = \mathbb{E}[X1_{Y \leq a}],$$

implying that $\mathbb{E}[(Y - X)1_{Y \leq a}] = 0$. Similarly, $\mathbb{E}[(X - Y)1_{X \leq a}] = 0$. Subtracting one from the other, we obtain

$$0 = \mathbb{E}[(X - Y)(1_{X \leq a} - 1_{Y \leq a})] = \mathbb{E}[(X - Y)(1_{X \leq a, Y > a} - 1_{X > a, Y \leq a})],$$

which we can write as

$$\mathbb{E}(X - Y)1_{X > a, Y \leq a} + \mathbb{E}(Y - X)1_{X \leq a, Y > a} = 0.$$  

Note that both $(X - Y)1_{X > a, Y \leq a} \geq 0$ and $(Y - X)1_{X \leq a, Y > a} \geq 0$. So,

$$(X - Y)1_{X > a, Y \leq a} = 0 \text{ and } (Y - X)1_{X \leq a, Y > a} = 0 \text{ a.s.}$$

This implies that

$$\mathbb{P}(X > a, Y \leq a) = 0 \text{ and } \mathbb{P}(X \leq a, Y > a) = 0 \text{ a.s.}$$

Finally,

$$\mathbb{P}(X > Y) = \mathbb{P}(X > a, Y \leq a \text{ for some } a \in \mathbb{Q}) \leq \sum_{a \in \mathbb{Q}} \mathbb{P}(X > a, Y \leq a) = 0.$$  

Similarly, $\mathbb{P}(Y > X) = 0$. Combining, we get $\mathbb{P}(X \neq Y) = 0$.

Q4. Let $X$ be integrable and let $\mathcal{G}, \mathcal{H}$ be two sub-$\sigma$-algebras of $\mathcal{F}$. If $\sigma(X, \mathcal{G})$ is independent of $\mathcal{H}$, prove that

$$\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}] \text{ a.s.}$$

[The notation $\sigma(\mathcal{G}, \mathcal{H})$ denotes the $\sigma$-field generated by $\sigma(\mathcal{X}) \cup \mathcal{G}$.]

Solution: We need to show that

$$\int_C \mathbb{E}[X|\mathcal{G}]dP = \int_C XdP$$

for any $C \in \sigma(\mathcal{G}, \mathcal{H})$. For any $A \in \mathcal{G}$ and $B \in \mathcal{H}$,

$$\int_{A \cap B} \mathbb{E}[X|\mathcal{G}]dP = \int \mathbb{E}[X|\mathcal{G}]1_A dPdP = \int \mathbb{E}[X|\mathcal{G}]1_A dP \times \int 1_B dP$$

$$= \int_A XdP \times \int 1_B dP = \int X1_A1_B dP = \int X1_{A\cap B} dP,$$

where the second equality above follows from the fact $\mathbb{E}[X|\mathcal{G}]1_A \in \mathcal{G}$ and $1_B \in \mathcal{H}$ and hence they are independent, the third equality is implied by the definition of $\mathbb{E}[X|\mathcal{G}]$ and the fourth equality is a consequence of the fact $X1_A$ and $1_B$ are independent since $X1_A$ is $\sigma(X, \mathcal{G})$-measurable while $1_B$ is $\mathcal{H}$-measurable.

Define

$$\mathcal{C} = \left\{ C \in \sigma(\mathcal{G}, \mathcal{H}) : \int_C \mathbb{E}[X|\mathcal{G}]dP = \int_C XdP \right\}.$$  

We have shown that $\mathcal{C}$ contains the $\pi$-system $\{A \cap B : A \in \mathcal{G}, B \in \mathcal{H}\}$. It is straightforward to see that $\mathcal{C}$ is also a $\lambda$-system (one can, for example, apply DCT to show that if $C_i \in \mathcal{C}$ and $C_i \uparrow$, then $\cup_i C_i \in \mathcal{C}$). So, by Dynkin’s $\pi - \lambda$ theorem, $\mathcal{C}$ contains the $\sigma$-algebra containing the $\pi$-system $\{A \cap B : A \in \mathcal{G}, B \in \mathcal{H}\} \supseteq \mathcal{G} \cup \mathcal{H}$. Hence, $\mathcal{C} = \sigma(\mathcal{G}, \mathcal{H})$. 

2
Q5 Durrett 5.1.6

**Solution:** We consider the probability space \((\Omega := \{a, b, c\}, 2^\Omega, P)\) where \(2^\Omega\) is the set of all subsets of \(\Omega\) and \(P\) is the uniform measure on \(\{a, b, c\}\). Take \(\mathcal{F}_1 = \sigma(a)\) and \(\mathcal{F}_2 = \sigma(b)\) and the random variable \(X = 1_{\{a\}}\). Note that \(E[\cdot|\mathcal{F}_1]\) is averaging over \(b, c\) and \(E[\cdot|\mathcal{F}_2]\) is averaging over \(a, c\). We compute

\[
E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[1_{\{a\}}|\mathcal{F}_2] = \frac{1}{2}1_{\{a\}} + \frac{1}{2}1_{\{c\}}
\]

and

\[
E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[\frac{1}{2}1_{\{a\}} + \frac{1}{2}1_{\{c\}}|\mathcal{F}_1] = \frac{1}{2}1_{\{a\}} + \frac{1}{4}1_{\{b\}} + \frac{1}{4}1_{\{c\}}.
\]

So, clearly \(E[E[X|\mathcal{F}_1]|\mathcal{F}_2] \neq E[E[X|\mathcal{F}_2]|\mathcal{F}_1]\).

Q6. 5.1.9

**Solution:** Note that

\[
E(X - E[X|\mathcal{F}])^2 = E[X^2] - 2E[XE[X|\mathcal{F}]] + E(E[X|\mathcal{F}]^2) = E[X^2] - E(E[X|\mathcal{F}]^2) = \text{EVar}(X|\mathcal{F}),
\]

where in the second equality we have used the fact that \(EXE[X|\mathcal{F}] = \text{EVar}(X|\mathcal{F})\).

Now

\[
\text{Var}(X) = E(X - EX)^2 = E(X - E(X|\mathcal{F}) + E(X|\mathcal{F}) - EX)^2
\]

\[
= E(X - E(X|\mathcal{F}))^2 + E(E(X|\mathcal{F}) - EX)^2
\]

\[
= E(X - E(X|\mathcal{F}))^2 + E(E(X|\mathcal{F}) - E(X|\mathcal{F}))^2
\]

\[
= \text{EVar}(X|\mathcal{F}) + \text{Var}(E(X|\mathcal{F}))
\]

where the cross term disappear because

\[
E[(X - E(X|\mathcal{F}))(E(X|\mathcal{F}) - EX)] = EE[(X - E(X|\mathcal{F}))(E(X|\mathcal{F}) - EX)]|\mathcal{F}
\]

\[
= E[(E(X|\mathcal{F}) - EX)E[(X - E(X|\mathcal{F})]|\mathcal{F}]
\]

\[
= E[(E(X|\mathcal{F}) - EX)(E(X|\mathcal{F}) - E(X|\mathcal{F}))] = 0.
\]

Q7. 5.1.11 **Solution:** From Q6, we get

\[
\]

which implies that \((Y - X)^2 = 0\) a.s. or, equivalently, \(X = Y\) a.s.