1. Recall the following version of Cauchy-Schwarz inequality that we proved in class: For any two random variables $X$ and $Y$, we have

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}.$$  

Use the above inequality to prove the following two versions of Cauchy-Schwarz inequality.

(i) For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, show that

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 \right)^{1/2}.$$  

[Hint: Take $X,Y$ to be discrete random variables with joint p.m.f. $f_{X,Y}(a_i, b_i) = \frac{1}{n}$ for all $i$.]

(ii) For any $a < b$ and any two bounded functions $f, g : [a,b] \to \mathbb{R}$, show that

$$\left| \int_{a}^{b} f(x)g(x)dx \right| \leq \left( \int_{a}^{b} f(x)^2dx \cdot \int_{a}^{b} g(x)^2dx \right)^{1/2}.$$  

2. Let $X$ and $Y$ be two i.i.d. exp(1) random variables. Recall that p.d.f. of an exp(1) random variable is given by

$$f(x) = e^{-x} \text{ if } x > 0 \text{ and } = 0 \text{ if } x \leq 0.$$  

Let $L = X - Y$. Our goal is to show that the p.d.f. of $L$ is

$$g(x) = e^{-\frac{1}{2}|x|}, x \in \mathbb{R}.$$  

Use the following two ways to prove the above claim.

(i) First compute the CDF of $L$ and then differentiate the CDF to obtain the p.d.f. of $L$.

(ii) Compute the moment generating function of $X - Y$ and show that it matches with moment generating function of any random variable whose p.d.f. is $g$.

[Correction (4/13): The above expression of $g(x)$ is incorrect. The correct one should read as $g(x) = \frac{1}{2}e^{-|x|}$.]

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with mean $\mu$ and variance $\sigma^2$ and the moment generating function $\psi$, that is, $\psi(t) = E[e^{tX_i}]$ for $t \in \mathbb{R}$. Let

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i , \quad \text{and} \quad Z_n = \sqrt{n} \left( \frac{\bar{X}_n - \mu}{\sigma} \right).$$  

(i) Show that $Z_n$ has mean 0 and variance 1.

(ii) Compute the moment generating function of $Z_n$ in terms of $\psi$.  

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