2.1. 4. Let $A_i$ = the shopper purchases brand $A$ on his $i$th purchase.

$$B_i = \frac{1}{n} B$$

To find $P(A_1 \cap A_2 \cap B_3 \cap B_4)$

By multiplication rule, $P(A_1 \cap A_2 \cap B_3 \cap B_4) = P(A_i) P(A_2 | A_1) \times P(B_3 | A_1 \cap A_2) \times P(B_4 | A_1 \cap A_2 \cap B_3)$.

Given: $P(A_i) = \frac{1}{2}$, $P(A_2 | A_1) = \frac{1}{3}$, $P(B_3 | A_1 \cap A_2) = \frac{2}{3}$, $P(B_4 | A_1 \cap A_2 \cap B_3) = \frac{1}{3}$.

(Since prob. of switch = $\frac{2}{3}$)

Hence, $P(A_1 \cap A_2 \cap B_3 \cap B_4) = \frac{1}{27}$.

6. Let $A$ = the observed side of the card is green and $B_i$ = $i$th card is selected.

(1st card = both red, 2nd card = both green, 3rd card = one red, other green)

$$P(B_2 | A) = \frac{P(A \cap B_2)}{P(A)} = \frac{P(B_2)}{\sum_{i=1}^{3} P(A | B_i) P(B_i)}$$

$$= \frac{1}{3} \quad \quad 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}$$

4. Let $D$ = the part is defective.

$A_1$ = the machine is in good working conditions.
$A_2 =$ the machine is wearing down
$A_3 =$ needs maintenance.

$P(A_1) = 0.8, \ P(A_2) = 0.1, \ P(A_3) = 0.1$
$P(D|A_1) = 0.02, \ P(D|A_2) = 0.1, \ P(D|A_3) = 0.3$.

$P(D) = \sum_{i=1}^{3} P(D|A_i) P(A_i) = 0.056$

2.2 Let $C_1 =$ the event that A attends the class, $C_2 =$ B.

Since $C_1$ & $C_2$ are independent, we have

$P(C_1 \cap C_2) = P(C_1) P(C_2)$.

a. $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$

$= P(C_1) + P(C_2) - P(C_1) P(C_2)$

$= 0.8 + 0.6 - 0.8 \times 0.6 = 0.92$

b. $P(C_1 | C_1 \cup C_2) = \frac{P(C_1 \cap (C_1 \cup C_2))}{P(C_1 \cup C_2)} = \frac{P(C_1)}{P(C_1 \cup C_2)}$

0. Let $A_j =$ the event that exactly $j$ children in the family have blue eyes.

$P(A_j) = \binom{5}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{5-j} = p_j$ (say)

[Binomial probability: same as $j$ heads in 5 independent tosses with head prob. $\frac{1}{4}$]
We want
\[
P(\underbrace{A_3 U A_4 U A_5} \mid \underbrace{A_1 U A_2 U \ldots U A_5})
\]
at least 3 blue eyes
\[
= \frac{P(A_3 U A_4 U A_5)}{P(A_1 U \ldots U A_5)} = \frac{P_3 + P_4 + P_5}{1 - P(A_0)} = \frac{P_3 + P_4 + P_5}{1 - P_0}
\]
\[
= \frac{(\frac{5}{9}) \frac{3^2}{4^5} + (\frac{5}{4}) \frac{3}{4^5} + \frac{1}{4^5}}{1 - (\frac{3}{4})^5}
\]
\[
= \frac{90 + 15 + 1}{4^5 - 3^5} = \frac{106}{781}
\]

18. let \( C_j \) = both \( A \) and \( B \) miss on each of their first \( j-1 \) throws, and then \( A \) hit the target on his next (\( j \)th) throw.

Note \( \{ \text{A hits the target before B} \} \)
\[
= \bigcup_{j=1}^{\infty} C_j \quad \text{&} \quad C_j's \text{ are disjoint.}
\]

\[
P(A \text{ hits before } B) = \sum_{j=1}^{\infty} P(C_j) = \sum_{j=1}^{\infty} \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{j-1}
\]
\[
= \frac{1}{3} \cdot \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1}{3} \cdot 2 = \frac{2}{3}.
\]
Alternative Solution: [borrowing similar ideas from Gambler's ruin problem]

\[ P(\text{A wins before B}) = P(\text{A hits before B} \mid \text{A hits on his 1st throw}) \]

\[ = P(\text{A hits before B} \mid \text{A hits on his 1st throw}) \cdot P(\text{A hits on his 1st throw}) \]

\[ + P(\text{A hits before B} \mid \text{A misses on his 1st throw, B hits on } ...) \cdot P(...) \]

\[ + P(\text{A hits before B} \mid \text{A misses on his 1st throw, B hits on } ...) \cdot P(...) \]

\[ = 1, \frac{1}{3} + 0.2\cdot \frac{1}{4} + a \cdot \frac{2}{3}, \frac{3}{4} \]

[When both A & B miss on their 1st throws, in effect, their starting a new game from the beginning]

So, \[ a = \frac{1}{3} + \frac{1}{2}a \Rightarrow a = \frac{2}{3} \]

Sec 2.5. \[ P(\text{A|D}) \geq P(\text{B|D}) \]

or, \[ \frac{P(\text{A AND D})}{P(D)} \geq \frac{P(\text{B AND D})}{P(D)} \]

or, \[ P(\text{A AND D}) \geq P(\text{B AND D}) \]
similarly \[ P(A \mid D^c) \geq P(B \mid D^c) \]
\[ P(A \cap D^c) \geq P(B \cap D^c) \]

\[ 0 + 0 \Rightarrow P(A \cap D) + P(A \cap D^c) \geq P(B) \]

Hence, \[ P(A) \geq P(B) \].