Math 2263
Quiz 12

Name
Section
Score

(8 points) Let \( F(x, y) = -\frac{y}{x^2+y^2}i + \frac{x}{x^2+y^2}j \).
(a) Show that \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).
(b) Show that \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is NOT independent of path. Does this contradict Theorem 6? [Hint: Compute \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \) and \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \), where \( C_1 \) and \( C_2 \) are the upper and lower halves of the circle \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((-1, 0)\)].

Theorem 6: Let \( \mathbf{F} = Pi + Qj \) be a vector field on an open simply-connected region \( D \). Suppose \( P \) and \( Q \) have continuous first-order derivatives and \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) throughout \( D \), then \( \mathbf{F} \) is conservative.

Solution: (a) \( P = -\frac{y}{x^2+y^2}, \frac{\partial P}{\partial y} = -\frac{x^2-y^2}{(x^2+y^2)^2} \), and \( Q = \frac{x}{x^2+y^2}, \frac{\partial Q}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \). Thus, \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).
(b) \( C_1: x = \cos t, y = \sin t, 0 \leq t \leq \pi; C_2: x = \cos t, y = \sin t, t = 2\pi \) to \( t = \pi \). Then
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-\sin t)(-\sin t) + (\cos t)(\cos t) \, dt = \int_0^\pi \, dt = \pi,
\]
and
\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi \, dt = -\pi.
\]
Since these aren’t equal, the line integral of \( \mathbf{F} \) isn’t independent of path. This doesn’t contradict Theorem 6, since the domain of \( \mathbf{F} \), which is \( \mathbb{R}^2 \) except the origin, isn’t simply-connected.

(7 points) Let \( D \) be a region bounded by a simple closed path \( C \) in the \( xy \)-plane. Use Green’s Theorem to prove that the coordinates of the centroid \((\bar{x}, \bar{y})\) of \( D \) are
\[
\bar{x} = \frac{1}{2A} \oint_C x^2 \, dy, \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 \, dx
\]
where \( A \) is the area of \( D \).

Solution: By Green’s theorem, \( \frac{1}{\pi A} \oint_C x^2 \, dy = \frac{1}{\pi A} \iint_D 2x \, dA = \frac{1}{\pi A} \iint_D x \, dA = \bar{x} \), and \( -\frac{1}{\pi A} \oint_C y^2 \, dx = -\frac{1}{\pi A} \iint_D -2y \, dA = \frac{1}{\pi A} \iint_D y \, dA = \bar{y} \).