Reduction of Order of Hamiltonian Systems
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Introduction

This is a paper on reduction of order of Hamiltonian systems using first integrals, where by order we mean the number of independent variables in the system. In the Hamiltonian case, Noether’s theorem (to be proved below) says that first integrals are equivalent to symmetries of the system and, hence, we could use Sophus Lie’s standard order reduction techniques to accomplish this. The story is, however, more interesting than that: Hamiltonian systems live on manifolds endowed with an additional structure of a Poisson bracket, and exploiting this extra structure will allow us to reduce the order of the system by twice the dimension of our symmetry group while still preserving the Hamiltonian nature of the problem. This is analogous to the case of a system of Euler-Lagrange equations (where the extra structure exploited is of variational nature).

The brother part of the paper exhibits the (easier) case of a system with one known first integral and how we can use it to reduce order by two. The reduction technique in this case uses the same key idea underlying the proof of the important Darboux’ theorem, which we give.

The last section is on multi-dimensional symmetry groups, and knowledge of more sophisticated mathematics is needed to fully understand the theory (in particular, Stephen Smale’s Momentum map plays an important role). We shall routinely mention manifolds in this section to make the exposition slicker, but the reader can always think of them as open subsets of Euclidean space. Fortunately, all we need to implement the ideas is basic calculus. Nevertheless, more is required of the reader of the final section while only basic knowledge of Hamiltonian systems and Lie symmetries is assumed for sections leading up to it.

First Integrals

Consider a Hamiltonian system in local coordinates \((x^1, \ldots, x^m)\)

\[ x' = J(x) \nabla H(x, t), \] (1)

where \(J\) is the structure matrix for a Poisson bracket (i.e. \(J_{ij} = \{x^i, x^j\}\)). The right hand side of (1) is equal to \(\{x, H\}\), and denoted by \(\vec{v}_H\). For any two functions \(F\) and \(G\), we have the relation \(\vec{v}_F(G) = \{G, F\} = -\{F, G\} = \vec{v}_G(F)\). A first integral for this system is simply a function \(P(x, t)\) that is constant along solutions to (1). There is an
immediate characterization of first integrals in terms of the Poisson bracket (let \( x(t) \) be the solution to (1)):
\[
\frac{d}{dt} P(x(t), t) = \frac{\partial P}{\partial t} + \vec{v}_H(P(x(t), t)) = \frac{\partial P}{\partial t} + \{P, H\},
\]
so \( P \) is a first integral if and only if \( \frac{\partial P}{\partial t} + \{P, H\} = 0 \) everywhere. Obviously, if \( P \) is a first integral, the Hamiltonian vector field associated with \( \frac{\partial P}{\partial t} + \{P, H\} = 0 \) is zero. On the other hand, this vector field is
\[
\frac{\partial \vec{v}_P}{\partial t} + [\vec{v}_H, \vec{v}_P],
\]
where \( \frac{\partial \vec{v}_P}{\partial t} \) is the Hamiltonian vector field associated with the function \( \frac{\partial P}{\partial t} \). (Here we have used the fundamental fact (easily proven using the Jacobi identity) that \( \vec{v}_{(F,G)} = [\vec{v}_G, \vec{v}_F] \), which underlies much of the theory of Poisson manifolds). The condition that the vector field (2) is zero, is precisely the condition that \( \vec{v}_P \) (in its evolutionary form) is an infinitesimal generator of a symmetry of (1). We have proved the first half of Noether’s theorem:

**Theorem 1.** A Hamiltonian vector field \( \vec{w} = \vec{v}_P \) is a symmetry for the system (1) if and only if there is a distinguished function \( C \) such that \( \tilde{P} = P + C \) is a first integral for the system.

**Proof.** We have already proved that if \( P \) is a first integral, then \( \vec{v}_P \) is a symmetry (take \( C = 0 \)). Conversely, let \( \vec{v}_P \) be a symmetry. This means that
\[
\frac{\partial \vec{v}_P}{\partial t} + [\vec{v}_H, \vec{v}_P] = 0.
\]
As demonstrated above, this is a Hamiltonian vector field associated with the function \( \frac{\partial P}{\partial t} + \{P, H\} \). Any function having trivial Hamiltonian vector field, must be a distinguished function, so we write
\[
\frac{dP}{dt} = \frac{\partial P}{\partial t} + \{P, H\} = \tilde{C}, \text{ distinguished.}
\]
Now define \( C(x, t) = \int_0^t \tilde{C}(x, \tau) d\tau \). The function \( C \) is also distinguished, and
\[
\frac{d}{dt} C(x(t), t) = \frac{\partial C}{\partial t} + \vec{v}_H(C(x(t), t)) = \tilde{C},
\]
so setting \( \tilde{P} = P - C \) proves the theorem. \( \square \)
One Dimensional Reduction

Any story about reduction of order necessarily begins with the statement of the powerful Frobenius theorem from differential geometry, see [1] for a proof. The theorem says when systems of vector fields can be "straightened out". Before we state the theorem we need a definition.

**Definition 1.** A system of vector fields \( \{ \vec{v}_1, \ldots, \vec{v}_r \} \) on a manifold are said to be in *involution* if the system preserves the commutator of its elements:

\[
[\vec{v}_i, \vec{v}_j] \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_r\}, \text{ for any } i,j.
\]

**Theorem 2.** If an \( r \) dimensional system of vector fields on an \( m \) dimensional manifold \( (r \leq m) \) is in involution, then we can choose local coordinates, \( (x^1, \ldots, x^m) \), in which the system becomes

\[
\{ \partial_{x^1}, \ldots, \partial_{x^r} \}.
\]

This is not the most general version of the theorem, but is sufficient for our purposes.

In the business of simplifying geometrical objects on manifolds, the Frobenius theorem plays the lead role. There is a nice corollary to this theorem: A function \( \xi \) is said to be *invariant* under the system of vector fields if it is constant along all its paths. We can then see from the Frobenius theorem that an \( r \) dimensional system on an \( m \) dimensional manifold has exactly \( m-r \) functionally independent invariants. In the flat coordinate system \( (x^1, \ldots, x^m) \) provided by the Frobenius theorem, these functions are the component functions \( \xi^1(x) = x^{r+1}, \ldots, \xi^{m-r}(x) = x^m \).

Throughout the discussion, for simplicity, let us restrict ourselves to time-independent first integrals. Let \( \vec{v}_P \) be a Hamiltonian symmetry of (1), with \( P(x) \) a first integral. Just like when using Lie’s method of order reduction, we would like to "straighten out" the vector field \( \vec{v}_P \), meaning, changing variables \( (x^1, \ldots, x^m) \rightarrow (p, q, y_1, \ldots, y^{m-2}) \) such that \( \vec{v}_P = \partial_q \) in the new variables. Having achieved this, we will see that our system (1) transforms into simpler form, effectively reducing its order by two!

**Theorem 3.** Let \( \vec{v}_P \) be a Hamiltonian symmetry for (1), with \( P \) a time-independent first integral. The solution to (1) can be obtained from the solution to a Hamiltonian system in \( m-2 \) variables by quadratures.

*Proof.* Choose new variables \( x \rightarrow (P(x), Q(x), \xi^1(x), \ldots, \xi^{m-2}(x)) = (p, q, y_1, \ldots, y^{m-2}) \), guaranteed to exist by Frobenius, such that \( \vec{v}_P = \partial_q \). The structure matrix \( J \), in the
new variables becomes
\[
J = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & a \\
0 & a^T & \tilde{J}
\end{pmatrix},
\] (3)
where \( \tilde{J}_{ij}(p,q,y) = \{y^i,y^j\} \) and \( a(p,q,y) \) is an \( m - 2 \) dimensional row vector with elements \( \{q,y^i\} \). We shall prove that \( \tilde{J}, a \) and \( H \) are all independent of \( q \):

\[
\frac{\partial \tilde{J}_{ij}}{\partial q} = \{\{y^i,y^j\},p\} = -\{\{p,y^i\},y^j\} - \{\{y^i,p\},y^j\}
= \{\partial_q(y^i),y^j\} - \{\partial_q(y^j),y^i\} = 0.
\]

It is easy to see that, for a fixed \( p \), \( \tilde{J} \) is a structure matrix for a Poisson bracket on \( (y^1, \ldots, y^{m-2}) \). The vector \( a \) is independent of \( q \):

\[
\partial_q(a^i) = \partial_q(\{q,y^i\}) = \{\{q,y^i\},p\} = 0,
\]
using the Jacobi identity and the fact that \( \{p,q\} \) is constant. The function \( H \) is independent of \( q \):

\[
\partial_q(H) = \{H,p\} = \{H,P\} = 0,
\]
since \( P \) is a first integral. The Hamiltonian system (1), in the new variables is

\[
\frac{dp}{dt} = 0 \\
\frac{dq}{dt} = -\partial_p(H) + \sum_{j=1}^{m-2} a^j(p,y)\partial_{y^j}(H) \\
\frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}_{ij}(p,y)\partial_{y^j}(H).
\]

We knew \( p \) was going to be constant as indicated by the first equation. The third equation is, as promised, a Hamiltonian system of order two less that the original. Having solved the third equation for a fixed \( p \), we can integrate the second equation, as the left hand side does not depend on \( q \). \( \square \)

The above proof can be used, with minor modifications to prove Darboux’ theorem: Let \( J(x) \) be a structure matrix of rank \( 2n \), \( n \geq 1 \) and assume that any structure matrix of rank \( 2(n - 1) \) can be turned into the canonical one by a change of variable. Take any \( P(x) \) with \( \tilde{v}_p \neq 0 \) and change variables \( x \rightarrow y \), making \( \tilde{v}_p = \partial_{y^1} \). This tells us that there is a function \( Q(x) \) such that \( \tilde{v}_p(Q) \equiv 1 \) (in the new variables \( Q(y) = y^1 \)).
The vector fields $\vec{v}_P$ and $\vec{v}_Q$ are in involution since

$$0 = \vec{v}_1 = \vec{v}_{\vec{v}_P(Q)} = \vec{v}_{(Q,P)} = [\vec{v}_P, \vec{v}_Q].$$

Now choose new variables $x \rightarrow (P(x), Q(x), \xi^1(x), \ldots, \xi^{2(n-1)}(x)) = (p, q, y)$, such that $\vec{v}_P = \partial q$ and $\vec{v}_Q = \partial_p$ (again, these are guaranteed by the Frobenius theorem). Now the structure matrix has the same form as (3), except $a = 0$ and $\tilde{J}$ is also independent of $p$. By the induction hypothesis, $\tilde{J}(y)$ can be made canonical, proving the theorem.

Let us look at an example.

**Example 1.** Consider the canonical Poisson bracket on $\mathbb{R}^4$ and the Hamiltonian

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1 - q_2),$$

describing the motion of two unit point masses on a line, interacting via the potential $V$ that is only dependent on the distance between them. This system has symmetry $\vec{v} = \partial_{q_1} + \partial_{q_2}$, which is Hamiltonian corresponding to the first integral $p_1 + p_2$. The symmetry reflects the obvious fact that the system is invariant under simultaneous shifts in position. Now, how do we straighten out $\vec{v}$? We need four, independent variables, three that $\vec{v}$ annihilates and one that $\vec{v}$ turns into the constant 1. Looking at $\vec{v}$, we find these variables by trial and error (of course there are rigorous ways to doing this, but the present example is easy enough for us to do this by staring at it.) First, $p := p_1 + p_2$, being the first integral is always constant along $\vec{v}$. The function $q := q_1$ is obviously turned into a constant by $\vec{v}$. Two more that vanish along $\vec{v}$

$$y = p_1, \quad r = q_1 - q_2.$$

In the new variables $(p, q, y, r)$, our Hamiltonian is $H = y^2 - py + \frac{1}{2}p^2 + V(r)$, and the system is

$$\frac{dp}{dt} = 0, \quad \frac{dq}{dt} = y, \quad \frac{dy}{dt} = -V'(r), \quad \frac{dr}{dt} = 2y - p. \quad (4)$$

Remember that keeping $p$ fixed, we should have a Hamiltonian system in the $(y, r)$ variables, to find the structure matrix on this sub-manifold we calculate

$$\{y, r\} = \{p_1, q_1 - q_2\} = -1,$$

so the structure matrix is the canonical one

$$\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.$$
Related to the Darboux-type techniques covered so far, there is a trick to reduce the order of any time-independent Hamiltonian system by two. In this case, the Hamiltonian itself is a first integral and we make use of this fact. The drawback is that the reduced Hamiltonian system will be time-dependent. We continue as in the proof of Darboux, straightening out any two non-zero Hamiltonian vector fields $\vec{v}_P = \partial_q$ and $\vec{v}_Q = \partial_p$. This is accomplished by finding (functionally) independent $\xi^1(x), \ldots, \xi^{m-2}(x)$ such that $\vec{v}_P(\xi^i) = \vec{v}_Q(\xi^i) = 0$. In these variables the Hamiltonian system is

$$\frac{dp}{dt} = -\partial_q(H), \quad \frac{dq}{dt} = \partial_p(H), \quad \frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}(y)\partial_{y^j}(H).$$

Now, here is the key idea: Since $H$ is itself a first integral we make the following clever change of variables

$$(t, p, q, y) \rightarrow (q, t, w = H(p, q, y), y),$$

where $q$ is the new independent variable (time). Actually, for this transformation to be of full rank, we need $\partial_q(H) \neq 0$, so we assume this. Notice that if $\partial_p(H) = 0$, then $Q$ is a first integral (remember that $\vec{v}_Q = \partial_p$) and we can use the previous reduction result. In the new variables $(q, t, w, y)$ the system becomes (note that $\frac{dw}{dq} = \frac{dw}{dt} = 0$)

$$\frac{dt}{dq} = \frac{1}{\partial_p(H)}, \quad \frac{dw}{dq} = 0, \quad \frac{dy^i}{dq} = \sum_{j=1}^{m-2} \tilde{J}_{ij}(y)\frac{\partial_{y^j}(H)}{\partial_p(H)},$$

where $\tilde{J}$ is the structure matrix on the variables $(y^1, \ldots, y^{m-2})$. Now, since $\partial_p(H) \neq 0$, we can solve the equation $w = H(p, q, y)$ for $p = K(q, w, y)$ and differentiating on both sides of $w = H(K(q, w, y), q, y)$ w.r.t. $y^i$ we find that

$$0 = \partial_p(H)\partial_{y^i}(K) + \partial_{y^i}(H) \Leftrightarrow \partial_{y^i}(-K) = \frac{\partial_{y^i}(H)}{\partial_p(H)},$$

and hence, the equation for $y$ in (5) is Hamiltonian with Hamiltonian function $-K$. After solving this reduced system for a fixed $w$ we can solve the remaining equation for $t(q)$ by a quadrature.

Note that the above procedure is almost the same as the "naive" reduction of first restricting to a level set of the first integral $w = H$ and then using the systems autonomy to invoke Lie’s reduction technique. However, the reduced system will not be Hamiltonian except if we use the Darboux-type variables.
Example 2. In Example 1 we reduced a fourth order system to a second order one in the \((y, r)\) variables, with Hamiltonian \(H = y^2 - py + \frac{1}{2}p^2 + V(r)\), where \(p\) is constant. This Hamiltonian is time-independent. Consider the level set \(H = \omega + \frac{1}{4}p^2\), solving this equation for \(y\) gives \(y = \frac{1}{2}p \pm \sqrt{\omega - V(r)}\) and the system (4) for \((y, r)\) becomes the integrable
\[
\frac{dr}{dt} = 2y - p = \pm \sqrt{\omega - V(r)}.
\]

Multi-Parameter Groups

When the Hamiltonian system (1) has more than one first integral the reduction theory is not so straight forward anymore. If we have \(r\) first integrals, we would like to, in keeping with the one dimensional case, to reduce the order of the system by \(2r\). Unfortunately that is not always possible. The most famous example of when this fails is that of the three body problem where the system is 18 dimensional and 9 first integrals are known, but it is not possible to use those to integrate the system completely. What fails is that the structure of the \(r\) dimensional symmetry group associated to the first integrals is “unfavorable” when restricted to a common level set of the first integrals. Namely, it may not be solvable:

Definition 2. An \(r\) dimensional Lie group \(G\) is solvable if there exist sub-groups \(G^1, \ldots, G^r\) of \(G\) such that \(G^i\) has dimension \(i\),
\[
G^1 \subset G^2 \subset \cdots \subset G^r,
\]
and for each \(i\), \(G^{i-1}\) is a normal sub-group of \(G^i\).

See 2 for a thorough discussion on Lie’s method of order reduction.

We will continue working with time-independent first integrals. But we shall further assume that the first integrals associated with a Hamiltonian system (1) are in involution. This condition on the Poisson bracket is analogous to that for vector fields, introduced earlier:

Definition 3. Hamiltonian functions \(P_1, \ldots, P_r\) are said to be in involution if there are constants \(c_{ij}^k\) such that
\[
\{P_i, P_j\} = \sum_{k=1}^{r} c_{ij}^k P_k.
\]

In this case the \(P_i\)s are said to generate a Hamiltonian action on the space on which the system lives. Further, if the \(P_i\)s are first integrals, generating the infinitesimal symmetries of (1) they are said to generate a Hamiltonian symmetry group of (1).
Note that since $\vec{v}_{\{F,G\}} = [\vec{v}_G, \vec{v}_F]$, this means that the Hamiltonian vector fields $\vec{v}_{P_i}$ are in involution with structure constants $-c_{kj}^i$. An important fact from Lie group theory is that for each such system of vector fields, there is a Lie group generating the action (see [2] for a proof.) It is the structure of this group (when restricted to a common level set of the first integrals) that will determine how effective the reduction procedure will be. We need the concept of the orbit of the action of a Lie group $G$ on a manifold $M$:

**Definition 4.** Let $G$ act on $M$. The orbit of the action through $x \in M$ is the set

$$G_x := \{g \cdot x : g \in G\}.$$  

(In our case of a Hamiltonian action the action is just movement along the Hamiltonian flows of the vector fields $\vec{v}_{P_1}, \ldots, \vec{v}_{P_r}$, and in the jargon of differential geometry, the orbits are the integral manifolds of the system of vector fields.)

**Definition 5.** The quotient manifold $M/G$ is the set of all orbits $G_x, x \in M$.

This set is automatically given a manifold structure by the Frobenius theorem: The invariants of the group action constitute a coordinate chart for $M/G$. That also tells us that $M/G$ is $m - s$ dimensional, where $s$ is the dimension of the orbits. Functions $F$ that are $G$-invariant on $M$ (i.e. $F(g \cdot x) = F(x)$ for all $g$ and $x$) naturally restrict to functions $\tilde{F}$ on $M/G$ and any function $\tilde{F}$ on $M/G$ has a natural $G$-invariant extension to $M$. When the action is Hamiltonian, the invariant condition is equivalent to $\vec{v}_{P_i}(F) = \{F, P_i\} = 0$. The reduction procedure consists projecting our system on $M$ (or $\mathbb{R}^m$ when working in local coordinates) onto the lower dimensional quotient manifold $M/G$ and try to recover the solution to the original problem on $M$ from the solution to the reduced system. As mentioned before, this is not possible except the symmetry group $G$ is solvable.

In the Hamiltonian case, the quotient manifold actually preserves the bracket structure:

**Theorem 4.** The quotient manifold $M/G$, of a Poisson manifold $M$ with bracket $\{\cdot, \cdot\}$, obtained from a Hamiltonian group action has a Poisson bracket, $\{\cdot, \cdot\}_{M/G}$ obtained from restricting $\{\cdot, \cdot\}$ to the sub-manifold $M/G$. Moreover, if the system (1), where $H : M \rightarrow \mathbb{R}$, has $G$ as a symmetry group then the system (1) naturally restricts to a Hamiltonian system on $M/G$ whose solutions are projections of those of (1).

**Proof.** Let $F$ and $K$ be $G$-invariant functions on $M$. In order for $\{F, K\}_M$ to restrict to the function $\{\tilde{F}, \tilde{K}\}_{M/G}$, $\{F, K\}$ needs to be $G$-invariant, indeed:

$$\vec{v}_{P_i}(\{F, K\}_M) = \{\{F, K\}, P_i\} = 0,$$
by the Jacobi identity and the fact that \( \{ F, P_i \} = \{ K, P_i \} = 0 \).

If \( G \) is a Hamiltonian symmetry group for (1), then \( H \) is \( G \)-invariant (\( \bar{v}_{P_i}(H) = \{ H, P_i \} = 0, P_i \) being a first integral.) Hence, \( H \) restricts to a function \( \tilde{H} \) on \( M/G \). To prove that the solutions to (1) project to those of the Hamiltonian system on \( M/G \) with Hamiltonian \( \tilde{H} \), we note that (let \( \pi \) be the smooth projection map \( M \to M/G \))

\[
d\pi(\bar{v}_H)(\tilde{F}) \circ \pi = \bar{v}_H(\tilde{F} \circ \pi) = \{ \tilde{F} \circ \pi, H \} \circ \pi,
\]

and so, \( d\pi(\bar{v}_H)(\tilde{F}) = \{ \tilde{F}, \tilde{H} \}_{M/G} = \bar{v}_{\tilde{H}}(\tilde{F}) \). Therefore \( \bar{v}_H \) is projected to \( \bar{v}_{\tilde{H}} \).

At this point we could charge on, reducing the order of a Hamiltonian system by Lie’s classical method, arriving at a Hamiltonian system on some quotient manifold and try to recover the solution to the original problem. While the reduced system is, by the last theorem, Hamiltonian, the group \( G \) has to be solvable for us to recover the original solutions. If the group has relatively high dimension it is very unlikely that it will be solvable. The approach presented below partially circumvents this annoying fact of nature by first restricting our system to a level set of the first integrals (remember that solutions are confined to these) and then working with the residual symmetry group acting on the level set, using it to reduce the order. Typically, this group is a proper subgroup of the whole group and therefore has a better chance of being solvable. In general, it is also much less work using small symmetry groups than large ones.

To see this, let \( P_1(x), \ldots, P_r(x) \) be first integrals for (1). As in the one dimensional case, we can restrict to a common level set of the integrals. Let \( \alpha \in \mathbb{R}^r \), and

\[
S_\alpha = \{ P_1(x) = \alpha_1, \ldots, P_r(x) = \alpha_r \}
\]

be the level set. Now we have this \( m - r \) dimensional system on \( S_\alpha \), but we cannot have the full group \( G \) with infinitesimal generators \( \bar{v}_{P_1}, \ldots, \bar{v}_{P_r} \) act on it. Only the elements of the subgroup that preserve \( S_\alpha \) can act on the reduced system:

\[
G_\alpha := \{ g \in G : g \cdot S_\alpha \subseteq S_\alpha \}.
\]

It is in determining the structure (solvability/non-solvability) of this subgroup of \( G \) that Stephen Smale’s (of the horseshoe) momentum map proves useful.\(^1\) The important result on these matters is the following:

**Theorem 5.** The quotient manifold \( S_\alpha/G_\alpha \) has a natural immersion into \( G/M \) as a Poisson sub-manifold, and Hamiltonian systems on \( M \) restrict to Hamiltonian systems

\(^1\)Smale’s result is that the group \( G_\alpha \) coincides with the *isotropy* subgroup of the co-adjoint representation of \( G \) on the dual to its Lie algebra: \( G_\alpha = \{ g \in G : \text{Ad}^* g(\alpha) = \alpha \} \).
on $S_\alpha/G_\alpha$. 

The interested reader can follow up on this theory in [2]. The theorem tells us that the fully reduced system $S_\alpha/G_\alpha$ is a Hamiltonian system in its own right and therefore susceptible to the techniques in the former sections. The theory is getting quite involved, fortunately the implementation of these ideas is simple.

Example 3. Let $M = \mathbb{R}^6$ with coordinates $(p, q) = (p^1, p^2, p^3, q^1, q^2, q^3)$ and the canonical Poisson bracket. Consider the two functions

$$P_1(p, q) = p^3, \quad P_2(p, q) = q^1 p^2 - q^2 p^1,$$

with corresponding Hamiltonian vector fields

$$\vec{v}_1 = \frac{\partial}{\partial q^3}, \quad \vec{v}_2 = p^1 \frac{\partial}{\partial p^2} - p^2 \frac{\partial}{\partial p^1} + q^1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial q^1}.$$

This system is in involution and generates a Hamiltonian group action on $M$ by an abelian two dimensional group $G$. An easy applications of Smale’s result is that the isotropy group $G_\alpha = G$, i.e. is the whole group spanned by $P_1$ and $P_2$. We are thus guaranteed that we can reduce the order of any Hamiltonian system having $G$ as a symmetry group by four. All Hamiltonians of the form $H(\rho, \sigma, \gamma, \zeta, t)$, where

$$\rho = \sqrt{(q^1)^2 + (q^2)^2}, \quad \sigma = \sqrt{(p^1)^2 + (p^2)^2}, \quad \gamma = P_1 \quad \text{and} \quad \zeta = P_2.$$

For example, the energy function $H = \frac{1}{2}|p|^2 + V(\rho)$ is of this kind.

Changing into polar coordinates $p = (\sigma \cos(\psi), \sigma \sin(\psi), \zeta)$, and $q = (\rho \cos(\theta), \rho \sin(\theta), z)$, we have $\vec{v}_1 = \partial_z$ and $\vec{v}_2 = \partial_\psi - \partial_\theta$. Writing the Hamiltonian system in the variables $(\gamma, \zeta, \theta, \phi = \psi - \theta, z)$, we find (besides $\gamma' = 0$, and $\zeta' = 0$)

$$\rho' = \cos(\phi)H_\sigma, \quad \phi' = \sin(\phi)(\sigma^{-1}H_\rho - \rho^{-1}H_\sigma),$$

$$\theta' = \rho^{-1}H_\sigma + H_\gamma, \quad z' = H_\zeta.$$

These variables were chosen to straighten out $\vec{v}_1$ and $\vec{v}_2$ as they are transformed to $\partial_z$ and $\partial_\theta$, respectively. This reduced system on $S_\alpha$ (where $\alpha$ is arbitrary) has $G_\alpha = G$ as a symmetry group, where $G$ is spanned by $\partial_z$ and $\partial_\theta$. We can see this in the fact that neither $z$ nor $\theta$ appear in the right hand sides of the reduced system. We can now use Lie’s classical reduction of order technique to finish the reduction. Notice that if our Hamiltonian is time-independent, we can reduce the fully reduced system yet by two and integrate this system completely.
References

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[2] Peter Olver
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