Divided differences and Newton’s formula

Notation: Let \( f \) be a real-valued function on an interval, and let \( x_0, \ldots, x_n \) denote \( n + 1 \) distinct points in \( J \). Let \( p_k \in P_k \) denote the Lagrange polynomial interpolating \( f \) at the first \( k + 1 \) points \( x_0, \ldots, x_k \).

1. Prove that
\[
p_n(x) - p_{n-1}(x) = c(x - x_0) \cdots (x - x_{n-1})
\]
for some constant \( c \). We use the notation \( f[x_0, \ldots, x_n] \) to denote this constant and call it the \( n \)th divided difference of \( f \) at the \( x_i \). Use Lagrange’s formula for the interpolating polynomial to derive an expression for \( f[x_0, \ldots, x_n] \) in terms of \( x_i \) and \( f(x_i) \).

2. Prove that \( f[x_0, \ldots, x_n] \) is a symmetric function of its \( n + 1 \) arguments.

3. Prove the recursion relation
\[
f[x_0, \ldots, x_n] = f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}],
\]
where, by convention, \( f[x] := f(x) \). (This explains the terminology “divided difference”.)

4. Give explicit formulas for \( f[a], f[a, b], f[a, b, c] \), and \( f[x, x + h, x + 2h, \ldots, x + nh] \).

5. Prove Newton’s formula for the interpolating polynomial
\[
p_n(x) = f[x_0] + f[x_0, x_1] (x - x_0) + f[x_0, x_1, x_2] (x - x_0) (x - x_1) \cdots
+ f[x_0, \ldots, x_n] (x - x_0) \cdots (x - x_{n-1}),
\]
and the error formula
\[
f(x) - p_n(x) = f[x_0, \ldots, x_n, x](x - x_0) \cdots (x - x_n).
\]

6. Prove that if \( f \in C^n(J) \), then there exists a point \( \xi \) in the interior of \( J \) such that
\[
f[x_0, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi).
\]

7. Assuming that \( f \in C^n(J) \), use the recursion defining the divided differences to establish the Hermite-Gennochi formula
\[
f[x_0, \ldots, x_n] = \int_{S_n} f^{(n)}(t_0 x_0 + t_1 x_1 + \cdots + t_n x_n) \, dt,
\]
where
\[
S_N = \left\{ \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{1}^{n} t_i \leq 1 \right\},
\]
and \( t_0 = 1 - \sum_{1}^{n} t_i \).

8. The Hermite-Gennochi formula shows that as a function of \( n + 1 \) variables the \( n \)th divided difference extends to a function on all of \( J^{n+1} \) (the arguments need not be distinct). Find simple closed form expressions for \( f[a, a], f[a, a, b], \) and \( f[a, a, b, b] \).