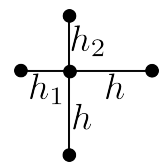
Math 8445 study problems

1. When solving the Poisson equation on a domain with curved boundary, we are led to seek an approximation of $\Delta u(x)$, based on the stencil shown to the right, where $h_1, h_2 < h$ (the Shortly–Weller formula). Derive the approximation formula for the stencil shown, and state an error bound for it.



2. Let Ω_h denote the grid $\{\,(ih, jh)\,|\,0 \le i, j \le n\,\}$ where $h = 1/n$. Prove that the grid function

$$\phi(x, y) = \sin(m\pi x)\sin(p\pi y), \quad (x, y) \in \Omega_h,$$

is an eigenfunction of $-\Delta_h$ and find the corresponding eigenvalue.

3. Suppose $A = P + Q$ where P is a tridiagonal matrix and Q has all zeros on the diagonal, subdiagonal, and superdiagonal. This leads to the iterative method

$$u_{i+1} = (1 - \alpha)u_i + \alpha P^{-1}(f - Qu_i).$$

a) What is the iteration matrix for this method?

b) Under what conditions on this matrix does the method converge, and at what rate of convergence?

4. Consider solving an SPD system $Au = f$ using the Richardson method with a parameter ω.

a) Derive the formula for the parameter giving the optimal rate of convergence.

b) What is the rate of convergence with this choice of parameter if the matrix has eigenvalues ranging from 10 to 100?

5. a) Write down the conjugate gradient iteration.

b) Prove that the search directions belong to the Krylov space generated by the matrix A and the initial residual r_0.

6. a) Write down the symmetric Gauss-Seidel iteration for solving a system $Ax = b$.

b) Viewing it as a residual correction method, what is the approximate inverse matrix?

c) Is the approximate inverse symmetric for (i) all A, (ii) just for symmetric A, or (iii) for neither in general? Justify.

7. What search direction and what step length are used in the method of steepest descents? Derive the formulas for both.

8. The damped Jacobi method is

$$u_{i+1} = (1 - \alpha)u_i + \alpha D^{-1}(f - (A - D)u_i).$$

Let $A = -D_h^2$ (the 3-point Laplacian in 1D) and consider the method for $\alpha = 1$ (standard Jacobi), $\alpha = .5$ (damped Jacobi), and $\alpha = 1.5$ (amplified Jacobi).

a) For which of these three values does the method converge?

b) For which of them does it have the smoothing property required by multigrid?

9. State the discrete maximum principle for the 5-point Laplacian and use it to prove stability.

10. Consider the basic finite difference method for solving

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is the unit cube in 3 dimensions, using a mesh spacing of $h = 1/n$.

a) How many rows in the resulting matrix?

b) How many nonzero elements are in the matrix? (Give the leading term in n, but you can ignore lower powers of n.)

c) The number of operations needed to do a banded LU or Cholesky factorization of this matrix grows as what power of n?

11. a) Given an SPD matrix $A \in \mathbb{R}^{n\times n}$ and a vector $f \in \mathbb{R}^n$, what is the quadratic function $F : \mathbb{R}^n \to R$ whose minimizer is the solution u of $Au = f$? Prove your assertion.

b) Show that for any $f \neq 0$, the minimum value of F is negative.

12. Consider the solution of the boundary value problem

$$-\Delta u + cu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is the unit square, c and f are continuous functions on Ω, and c is strictly positive.

Let $A = (a_{ij})$ by the $n \times n$ matrix resulting from discretizing this using the 5-point Laplacian with a grid of size $h = 1/(n + 1)$.

a) Show that the matrix A has positive diagonal elements and non-positive off-diagonal elements.

b) Show that the matrix A is strictly diagonally dominant: $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$.

c) Show that whenever a matrix A satisfies the conditions of a) and b), it satisfies the following property (related to the maximum principle):

If $Ax = b$ with $b_i \leq 0$ for all i, then $x_i \leq 0$ for all i.

13. Let A be a nonsingular $n \times n$ matrix, *not* necessarily symmetric or positive definite, and let u and f be vectors in $\mathbb{R}^n$. Show that u solves $Au = f$ if and only if u minimizes the functional

$$J(v) = \frac{1}{2}(Av)^T(Av) - f^T Av, \quad v \in \mathbb{R}^n.$$

14. Let Ω denote the unit square $(0, 1) \times (0, 1)$ and consider solving the boundary value problem

$$-\Delta u + u = 3x + 2y \text{ in } \Omega, \quad u = (1 - x)(1 - y) \text{ on } \partial\Omega,$$

using finite differences with a uniform mesh of mesh size $h = 1/n$. Let u_{ij} be the value of the numerical solution at grid point (ih, jh). For $n = 3$, the vector of unknowns is $U = (u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2})$. Find the 4×4 matrix A and the vector b, such $AU = b$. (It is not necessary to compute U.)

15. Consider the iteration $\quad x_{n+1} = (1 - 4\omega)x_n - 3\omega y_n + 10\omega, \quad y_{n+1} = (1 - 4\omega)y_n - 3\omega x_n + 11.$ For what values of the parameter ω does it converge?

b) What parameter values gives the fast convergence?

c) Assuming that ω is chosen so the iteration converges, write down a linear system of equations satisfied by the limit.