

LOCALIZATION OF EIGENFUNCTIONS VIA AN EFFECTIVE POTENTIAL

DOUGLAS N. ARNOLD, GUY DAVID, MARCEL FILOCHE, DAVID JERISON,
AND SVITLANA MAYBORODA

ABSTRACT. We consider the Neumann boundary value problem for an elliptic operator $L = -\operatorname{div} A \operatorname{grad} + V$ on a domain in \mathbb{R}^n that is the image of a bounded, smooth domain under a bi-Lipschitz mapping. More generally, we will treat closed manifolds and manifolds with boundary. The eigenfunctions of L are often localized, as a result of disorder of the potential V , the matrix of coefficients A , irregularities of the boundary, or all of the above. In earlier work, two of us introduced the function u solving $Lu = 1$, and showed numerically that it strongly reflects this localization. In this paper, we deepen the connection between the eigenfunctions and this *landscape* function u by proving that its reciprocal $1/u$ acts as an *effective potential*. The effective potential governs the exponential decay of the eigenfunctions of the system and delivers information on the distribution of eigenvalues near the bottom of the spectrum.

1. INTRODUCTION

Localization is a phenomenon in which eigenfunctions of an elliptic system concentrate on a small portion of the original domain and are nearly zero in the remainder, hindering and sometimes even totally preventing wave propagation. Over the past century it has been a source of wide interest in condensed matter physics and engineering, with an enormous array of applications. See, e.g., [Ab, An, AM, FS, BK].

In 2012, two of us introduced the concept of the *landscape*, namely the solution u to $Lu = 1$ for an elliptic operator L , and showed that this function has remarkable power to predict the shape and location of localized low energy eigenfunctions of L , whether the localization is triggered by the disorder of the potential, the geometry of the domain, or both (see [FM]). Since

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then, the landscape function has been used in theoretical and experimental physics to predict the vibration of plates, the spectrum of the bilaplacian with Dirichlet data [L+2016], the quantum droop and efficiency of GaN light emitting devices [FPW+], and the spectral properties of the Schrödinger operator with Anderson or Anderson-Bernoulli potential in bounded domains [A+2016].

The present paper offers the first mathematical treatment of the landscape function. We show that the reciprocal $1/u$ of the landscape function should be viewed as an *effective quantum potential*, in the sense that the eigenfunctions of L reside in the wells of $1/u$ and decay exponentially across the barriers of $1/u$. This implies that the landscape function often allows us to split the original domain into independently vibrating regions, and the spectrum of the original domain can be mapped bijectively on the combined spectrum of the subregions delimited by the barriers. In addition, the global eigenfunctions are exponentially close to eigenfunctions of subregions.

Our results are most useful for low eigenvalues and represent the first layer of what we expect to be a full decoupling, or diagonalization, of the operator modulo exponentially small errors. Numerically, we see even more separation and exponential decay than our results can guarantee. We prove decay between wells, but do not yet take into account resonance, namely, that eigenfunctions in separate wells only interact when the corresponding eigenvalues are exponentially close. Nevertheless, we believe this work is a substantial step towards explaining why, in practice, all low energy eigenfunctions are strongly localized in the subregions identified by $1/u$ [FM, A+2016, L+2016, FPW+].

In a complementary paper [A+2017], we explore how to use the landscape to predict spectral features numerically, without ever computing eigenfunctions or eigenvalues themselves. In particular, we demonstrate that the local minima of the effective potential, properly normalized, provide a good approximation to the eigenvalue distribution that is computationally efficient.

At this stage we want to issue a disclaimer. Although we have in mind the Anderson model as an example to which this paper applies, the paper does not address Anderson localization directly. Indeed, there are no theorems here concerning probability. What we prove are deterministic theorems showing that the shape of $1/u$ strongly influences the behavior of eigenfunctions and eigenvalues. Thus, by computing $1/u$ (as we mentioned just above, an easier task, numerically, than computing many eigenfunctions or eigenvalues) one can recognize Anderson localization. An appropriate probabilistic conjecture in the spirit of Anderson localization would say that almost surely for some family of potentials V , the effective potential $1/u$ has separated wells whose depths are nearly independent.

We summarize our main results in the very special case in which the operator is (minus) the ordinary Laplace operator plus a nonnegative bounded potential,

$$L = -\Delta + V,$$

acting on periodic functions, that is, on the manifold $M = \mathbb{R}^n/T\mathbb{Z}^n$. We will see that our estimates do not depend on the period T or the dimension n , only on the upper bound on the potential, \overline{V} , such that

$$0 \leq V(x) \leq \overline{V}.$$

In fact, our estimates are universal. In the body of the paper, we will treat operators with continuous coefficients on domains in \mathbb{R}^n that are bi-Lipschitz images of smooth domains. We will discuss extensions to manifolds in the last sections, Sections 5 and 6.

Assume that V is positive on a set of positive measure. Then the landscape function u , the solution to $Lu = 1$ on M , exists and is unique. Moreover, $u > 0$ by the maximum principle. Our starting point is the identity (Lemma 3.1)

$$(1.1) \quad \int_M [|\nabla f|^2 + Vf^2] dx = \int_M \left(u^2 |\nabla(f/u)|^2 + \frac{1}{u} f^2 \right) dx,$$

which holds for all $f \in W^{1,2}(M)$. In particular, if $\langle \cdot, \cdot \rangle$ denotes inner product on $L^2(M)$, then, since $u^2 |\nabla(f/u)|^2 \geq 0$,

$$(1.2) \quad \langle Lf, f \rangle \geq \langle (1/u)f, f \rangle.$$

This inequality is a form of the uncertainty principle, replacing the potential V in the trivial inequality $\langle Lf, f \rangle \geq \langle Vf, f \rangle$ with a new effective potential function $1/u$ that combines effects of the kinetic term $|\nabla f|^2$ and the potential term Vf^2 .

Inequalities of the form (1.2) are the key ingredient in the method of Agmon for estimating exponential decay of eigenfunctions. Roughly speaking, his theorem says that if (1.2) holds, then eigenfunctions of eigenvalue λ have “most” of their mass in the region

$$E(\lambda + \delta) = \{x \in M : 1/u(x) \leq \lambda + \delta\}$$

for a suitable small $\delta > 0$, and exponential decay in the complementary region. Thus, using Agmon’s method, we will be able to prove that we have localization provided the “well” $E(\lambda + \delta)$ consists of one or more small, localized regions. This is exactly what we are aiming for. Indeed, we don’t expect localized eigenfunctions if $E(\lambda + \delta)$ is not localized.

The most important aspect of the present work is the dramatic improvement in decay that arises from Agmon estimates using $1/u$ rather than V . The usual Agmon method confines eigenfunctions with eigenvalue λ to regions of the form $\{x \in M : V(x) \leq \lambda + \delta\}$. But this is nearly useless if V is disordered, for instance, if V is piecewise constant on unit cubes with random independent, identically-distributed values (either uniformly distributed between 0 and \overline{V} or Bernoulli, that is, taking the two values 0 and \overline{V}). This is especially true in the case of a Bernoulli potential where the 0-valued region percolates through the entire domain. In this situation, Agmon estimates provide no information about the exponential decay between

points belonging to the same percolating region, independently of their mutual distance. By contrast, in these random regimes, $1/u$ gives a remarkably clear separation into disjoint regions based on $E(\lambda + \delta)$, one that coincides with the actual behavior of eigenfunctions and changes appropriately as λ varies. (See [FM, A+2016, L+2016].)

To formulate our theorems more precisely, consider the weights

$$w_\lambda(x) := \max\left(\frac{1}{u(x)} - \lambda, 0\right).$$

Exponential decay is expressed in terms of the Agmon distance, a degenerate metric defined on M by

$$\rho_\lambda(x, y) = \inf_\gamma \int_0^1 w_\lambda(\gamma(t))^{1/2} |\dot{\gamma}(t)| dt,$$

with the infimum taken over absolutely continuous paths $\gamma : [0, 1] \rightarrow M$ from $\gamma(0) = x$ to $\gamma(1) = y$.

Let ψ be an eigenfunction: $L\psi = \lambda\psi$ on M . Let

$$h(x) = \inf\{\rho_\lambda(x, y) : y \in E(\lambda + \delta)\}$$

be the Agmon distance from x to $E(\lambda + \delta)$. The main decay estimate (Corollary 3.5) will be obtained by substituting $f = \chi e^h \psi$ in the identity (1.1), with χ a cutoff function that is 1 on $\{h \geq 1\}$ and zero on $E(\lambda + \delta)$. It says that

$$(1.3) \quad \int_{\{h \geq 1\}} e^h (|\nabla \psi|^2 + \bar{V} \psi^2) dx \leq C \int_M \bar{V} \psi^2 dx,$$

with

$$C = \frac{50 \bar{V}}{\delta}.$$

As the Agmon distance $h = h_{\lambda, \delta}$ from $E(\lambda + \delta)$ increases, the square density and energy of the eigenfunction are at most of size e^{-h} . We will apply this estimate to demonstrate an approximate diagonalization of the operator. In that proof it will be important to have exponential decay of $|\nabla \psi|^2$ in addition to the decay of $\bar{V} \psi^2$. Moreover, not surprisingly, in the proof of (1.3), to obtain decay of $|\nabla \psi|^2$, we will need not only the lower bound (1.2) but also the full identity (1.1).

The dependence of inequality (1.3) on V is expressed entirely in terms of \bar{V} and Agmon distance associated with the effective potential $1/u$. If one tracks the dependence of the quantities involved in (1.3) after the dilation replacing $\psi(x)$ by $r^{-2} \psi(rx)$ one finds, that T is replaced by T/r , $V(x)$ is replaced by $V(rx)$, and \bar{V} , λ , and δ are unchanged. Thus the constant C is unchanged under changes of scale of the region. Note also that C is independent of the dimension n . Later on we will see that it does not depend on the shape of the region or the ellipticity constants of the underlying operators.

We now turn to the approximate diagonalization of L . Fix $\bar{\mu}$, and consider any eigenvalue $\lambda \leq \bar{\mu} - \delta$. Choose any subdivision of $E = E(\bar{\mu} + \delta)$ into a finite collection of disjoint closed subsets

$$E = \bigcup_{\ell} E_{\ell}.$$

Let S denote the least Agmon distance relative to $\rho_{\bar{\mu}}$ between distinct pairs of sets E_{ℓ} and $E_{\ell'}$. One can visualize the E_{ℓ} as the connected components of E . In numerical examples, this is often the right choice, but because the result we will describe is stronger if the minimum separation S is large, it is sometimes better to merge two nearby components into a single set E_{ℓ} .

Next choose disjoint open sets $\Omega_{\ell} \supset E_{\ell}$ so that each Ω_{ℓ} lies at Agmon $\rho_{\bar{\mu}}$ distance at least $S/2$ from $E_{\ell'}$ for every $\ell' \neq \ell$. The sets Ω_{ℓ} are chosen in the spirit of Voronoi cells. Let $\varphi_{\ell,j}$, $j = 1, \dots$, be the orthonormal basis of $L^2(\Omega_{\ell})$ of eigenfunctions of L satisfying the Dirichlet condition $\varphi = 0$ on $M \setminus \Omega_{\ell}$. By results analogous to the exponential bounds for ψ , these functions $\varphi_{\ell,j}$ are concentrated near E_{ℓ} and decay exponentially in the typically much larger region Ω_{ℓ} . The union of these bases forms an orthonormal basis for $L^2(\bigcup \Omega_{\ell})$. Denote by $\Phi_{(a,b)}$ the orthogonal projection onto the subspace of $L^2(\bigcup \Omega_{\ell})$ spanned by eigenvectors with values between a and b . The main result (see Theorem 4.1) is that if ψ is a norm one eigenfunction for L with eigenvalue λ on M and $\lambda \leq \bar{\mu} - \delta$, then

$$\|\psi - \Phi_{(\lambda-\delta, \lambda+\delta)}\psi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta}\right)^3 e^{-S/2}.$$

Here we restricted to $M = \mathbb{R}^n/T\mathbb{Z}^n$ to simplify the exposition, but in Theorem 4.1 the constant is the same and does not depend on M . There is a similar estimate in which we take an eigenfunction $\varphi_{\ell,j}$ and show that it lies close to its orthogonal projection on a subspace spanned by a spectral band of eigenfunctions of the original operator L on all of M .

As a consequence (see Corollary 4.2), if $N_0(\lambda)$ denotes the cumulative eigenvalue counting function for the union of the $\varphi_{\ell,j}$ and $N(\lambda)$ denotes the counting function for the original operator L on M , we have

$$N_0(\lambda - \delta) \leq N(\lambda) \leq N_0(\lambda + \delta)$$

provided $\lambda \leq \bar{\mu} - \delta$ and these values are below the threshold \bar{N} given by

$$300\bar{N} \left(\frac{\bar{V}}{\delta}\right)^3 < e^{S/2}.$$

For instance, the first eigenvalue can be identified with a precision $\delta = 10\bar{V}e^{-S/6}$.

We performed numerical experiments in dimension $n = 1$ in order to estimate the size of the minimum well separation S as a function of the period T and found that it conforms well to a power law. For each value of T in 2^5 ,

$2^6, \dots, 2^{19}$, we computed 200 independent realizations of the random piecewise constant potential V on a period interval of length T . The constant values of V were chosen independently on T unit subintervals, uniformly distributed in the range from 0 to $\bar{V} = 4$. For each realization, we computed $1/u$ and the minimum separation S between consecutive connected components of $E(\lambda_0)$, that is, the set of wells of $1/u$ dipping below the smallest eigenvalue λ_0 of $-(d/dx)^2 + V$ with periodic boundary conditions on $[0, T]$. (These “wells” are typically unit length or smaller.) We then found the median of these 200 numbers S for each k . The best linear log-log fit for these medians, as a function of T , was $\text{median}(S) \sim .69 T^{.59}$, with correlation coefficient greater than 0.99. This shows that our theorems have content, simply because e^{T^α} grows faster than polynomially in T for $\alpha > 0$. On the other hand, the numerical evidence shows that the theorems do not capture the full phenomenon. For example, at $T = 2^{19}$, the median number of components of $E(\lambda_0)$ is 12. Our theorems show, roughly speaking, that the λ_0 (ground state) eigenfunction is a superposition of 12 localized eigenfunctions. Thus, we only prove that eigenfunctions are superpositions of spectral bands of localized eigenfunctions, a kind of block diagonalization. In fact, numerically, the ground state is most often supported in just one of these wells, and one can even tell from $1/u$ which one, the deepest.

To prove a result closer to what we actually see numerically, one must make further assumptions about the absence of resonance between putative eigenvalues in different wells. The assumptions would have to rule out periodic potentials, which do have this resonance and whose associated eigenfunctions can have significant mass in many wells.

There is a large literature about potential wells, exponential decay and resonance between wells. In addition to the seminal work of Agmon[A], we wish to mention the work of Helffer and Sjöstrand [H, HS] and closely related results of Simon [S1, S2] concerning so-called semiclassical behavior of the operator $-\hbar^2 \Delta + V$ with $V \in C^\infty(M)$, as $\hbar \rightarrow 0$. This work has a different character because the eigenfunctions are approximated by the harmonic oscillator based on the quadratic term in the Taylor expansion of V at minima. The case of nonsmooth potentials V was explored by Fefferman and Phong, who gave order of magnitude estimates of the lowest eigenvalue and spectral counting function. It is most convenient to use Shen’s interpretation in [Sh]. He introduced the maximal function

$$V^*(x) = \inf_{r>0} \left\{ \frac{1}{r} : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.$$

At the core of the work in [F, FP] is a version of the uncertainty principle, stated in [S] as

$$\int_{\Omega} [|\nabla f|^2 + V f^2] dx \geq c \int_{\Omega} V^* f^2 dx$$

for all $f \in C_0^\infty(\Omega)$. This is in the same spirit as our uncertainty principle, but, because the constant $c > 0$ can only be roughly estimated, their type

of inequality can only describe superpositions of eigenfunctions over a large band of frequencies. By contrast, the fact that we have a precise constant in (1.1) is very important. The estimate using V^* treats all eigenfunctions as collections of approximate bumps with an accuracy up to a constant factor, while $1/u$ is sensitive to the precise shape of low-energy eigenfunctions with nearly identical eigenvalues.

The paper is organized as follows. In Section 2 we give our main definitions and state some preliminary estimates on the landscape function and eigenfunctions. In Section 3, we derive all of our exponential decay estimates in the setting of domains in \mathbb{R}^n . We emphasize, once again, that for the subsequent application it is crucial to prove decay of $|\nabla\psi|^2$ as well as ψ^2 , and we reiterate that the constants in the theorems do not depend on the domain or the bi-Lipschitz constant. In Section 4, we deduce the block diagonalization into approximate eigenfunctions localized in individual wells. The procedure in Section 4 is somewhat different from the one in the work of Helffer and Sjöstrand [H, HS], for example, because we use duality and weak equations rather than rely on integration by parts and smoothness. In Section 5, we describe how to generalize our theorems to manifolds and prove the boundary regularity theorems stated in Section 2. The boundary Hölder regularity of eigenfunctions and the landscape function needed for the proof follows from a well known reflection argument. We also address the difficulty that the Agmon metric is only defined for continuous coefficient matrices A ; because our estimates are independent of the modulus of continuity, we are able to use a fairly straightforward procedure to approximate bounded measurable coefficient matrices by continuous ones. The present version makes the global assumption that domain Ω (as a manifold with boundary in a C^1 manifold) is the bi-Lipschitz image of the interior of a compact C^∞ manifold with C^∞ boundary. This global topological assumption is only needed for technical reasons, and we will explain in Section 6 how we hope to replace it with a more natural local assumption.

2. MAIN ASSUMPTIONS AND PRELIMINARY ESTIMATES

Let Ω be the image under a bi-Lipschitz mapping of \mathbb{R}^n to \mathbb{R}^n of a bounded C^∞ domain. (In Section 5, we will replace the ambient space \mathbb{R}^n with a compact C^1 manifold \widehat{M} . In Section 6, we will give an outline of a proposed proof that applies when the boundary is locally, rather than globally bi-Lipschitz.)

Set $M = \overline{\Omega}$, and let $m \in L^\infty(\Omega)$ be a real-valued density satisfying uniform upper and lower bounds

$$\frac{1}{C} \leq m(x) \leq C,$$

for some positive constant C . Let $A = (a_{ij}(x))_{i,j=1}^n$ be a bounded measurable, real symmetric matrix-valued function, satisfying the uniform ellipticity condition

$$(2.1) \quad \frac{1}{C}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

for some $C < \infty$. We define the elliptic operator L acting formally on real-valued functions φ by

$$L\varphi = -\frac{1}{m} \operatorname{div}(mA\nabla\varphi) + V\varphi = -\frac{1}{m} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(m a_{ij} \frac{\partial \varphi}{\partial x_j} \right) + V\varphi.$$

The operator L will always be used in the weak sense, defined as follows.

Definition 2.1. A function $\varphi \in W^{1,2}(\Omega)$ satisfies $L\varphi = f$ weakly on Ω (respectively, on $M = \overline{\Omega}$) if

$$(2.2) \quad \int_{\Omega} [(A\nabla\varphi) \cdot \nabla\eta + V\varphi\eta] m \, dx = \int_{\Omega} f\eta m \, dx$$

for every $\eta \in W_0^{1,2}(\Omega)$ (respectively, for every $\eta \in W^{1,2}(\Omega)$).

Here the space $W^{1,2}(\Omega) = W^{1,2}(M)$ is the usual Sobolev space, namely the closure of $C^1(M)$ in the function space with square norm given by

$$\int_{\Omega} (|\nabla\varphi|^2 + \varphi^2) \, dx.$$

The space $W_0^{1,2}(\Omega)$ is the closure in the same norm of the subspace $C_0^1(\Omega)$ of continuously differentiable functions that are compactly supported in Ω .

The weak equation on $M = \overline{\Omega}$ imposes, in addition to the interior condition, a weak form of the Neumann boundary condition on φ . If there is sufficient smoothness to justify integration by parts, then the Neumann condition can be written

$$\nu(x) \cdot A(x)\nabla\varphi(x) = 0, \quad x \in \partial\Omega,$$

with ν the normal to $\partial\Omega$. In fact, in the case of Lipschitz boundaries, the Neumann condition is valid almost everywhere with respect to surface measure on $\partial\Omega$ for suitable right hand sides f . But, we will only need the weak form, not this strong version of the boundary condition. (For now we confine ourselves to Neumann boundary conditions; we will say a few words about Dirichlet and mixed boundary conditions later.)

We assume further that V is non-degenerate in the sense that it is strictly positive on a subset of positive measure of Ω . By ellipticity of A and the fact that Ω is a connected, bounded bi-Lipschitz domain, we have the coercivity inequality

$$\int_M [(A\nabla\varphi) \cdot \nabla\varphi + V\varphi^2] m \, dx \geq c \int_M (|\nabla\varphi|^2 + \varphi^2) \, dx,$$

for some $c > 0$. In other words, the formal $L^2(M, m dx)$ inner product $\langle L\varphi, \varphi \rangle$ is comparable to the square of the $W^{1,2}(\Omega) = W^{1,2}(M)$ norm of φ . By the Fréchet–Riesz theorem (identifying a Hilbert space with its dual), this implies that for every $f \in L^2(M, m dx)$, there is a unique solution $v \in W^{1,2}(M)$ to the weak equation $Lv = f$ on M . The *landscape function* u is defined as the solution to

$$Lu = 1 \quad \text{weakly on } M.$$

In other words, u is the unique weak solution to the inhomogeneous Neumann problem with right hand side the constant 1.

Proposition 2.2. *Let V be nondegenerate and satisfy $0 \leq V \leq \bar{V}$ for some constant \bar{V} . Then the landscape function $u \geq 1/\bar{V}$ on M . Moreover $u \in C^\alpha(M)$ for some $\alpha > 0$.*

Proof. Consider the weak solution to $Lv = f$ on M for bounded measurable f . Hölder regularity of v at interior points of M follows from a version of the theorem of De Giorgi, Nash, and Moser (see Theorem 8.24, [GT]). Near each boundary point, one can define an “even” reflection of v that satisfies a uniformly elliptic equation in a full neighborhood; hence v is C^α up to the boundary for some $\alpha > 0$. This reflection argument is presented in the last section in the more general context of manifolds (see Proposition 5.1). In particular, $u \in C^\alpha(M)$.

Next, we prove a version of the maximum principle, namely that $v \geq 0$ provided $f \geq 0$. Since v is continuous, the set $\Omega^- = \{x \in \Omega : v(x) < 0\}$ is open. Since v minimizes

$$\int_{\Omega} ((A\nabla\varphi) \cdot \nabla\varphi + V\varphi^2 - 2f\varphi) m dx$$

among all $\varphi \in W^{1,2}(M)$, we have

$$\begin{aligned} & \int_{\Omega} ((A\nabla v) \cdot \nabla v + Vv^2 - 2fv) m dx \\ & \leq \int_{\Omega} ((A\nabla v_+) \cdot \nabla v_+ + Vv_+^2 - 2fv_+) m dx \end{aligned}$$

for $v_+(x) = \max(v(x), 0)$. Consequently,

$$\int_{\Omega^-} ((A\nabla v) \cdot \nabla v + Vv^2 - 2fv) m dx \leq 0.$$

Because $V \geq 0$ and $f \geq 0$, we have $Vv^2 - 2fv \geq 0$ on Ω^- . Therefore,

$$\int_{\Omega^-} (A\nabla v) \cdot \nabla v m dx \leq 0.$$

Since A is coercive, $\nabla v = 0$ a.e. on Ω^- , and v is a strictly negative constant on each connected component of Ω^- . If any such component is a proper subset of Ω , then the continuity of v contradicts the fact that $v \geq 0$ on $\Omega \setminus \Omega^-$. On the other hand, if $\Omega^- = \Omega$, then $v \equiv -a$, for some constant

$a > 0$. But in that case, $Lv = -aV$, which cannot equal $f \geq 0$. Thus, the only possibility is that Ω^- is empty.

Finally, to conclude proof of the proposition, consider u , the weak solution to $Lu = 1$ on M . Then

$$v = u - \frac{1}{\bar{V}} \quad \text{solves} \quad Lv = 1 - \frac{V}{\bar{V}} \geq 0.$$

Therefore, by the maximum principle, $v \geq 0$, and $u \geq 1/\bar{V}$. \square

By the bi-Lipschitz assumption on Ω and the Rellich-Kondrachov lemma, the inclusion mapping $W^{1,2}(M) \hookrightarrow L^2(M)$ is compact. Thus, by the spectral theorem for compact operators, there is a complete orthonormal system of eigenfunctions to the Neumann problem for L , that is, an orthonormal basis ψ_j of $L^2(M)$ such that $\psi_j \in W^{1,2}(M)$, and

$$L\psi_j = \lambda_j\psi_j \quad \text{weakly on } M.$$

The non-degeneracy of V implies that the eigenvalues λ_j are strictly positive,

We will compare these eigenfunctions to localized eigenfunctions of Dirichlet or mixed boundary value problems. Let K be a compact subset of M . Let U be a connected component of $M \setminus K$. We say that $L\varphi = f$ weakly on U if equation (2.2) holds for all test functions $\eta \in C^1(M)$ such that the support of η is contained in U . We will denote the closure of this set of test functions in the usual $W^{1,2}(\mathbb{R}^n)$ norm by $W_0^{1,2}(U)$. Formally, solutions to $L\varphi = f$ on U satisfy mixed boundary conditions

$$\varphi(x) = 0, \quad x \in K \cap \partial U; \quad \nu(x) \cdot A(x)\nabla\varphi(x) = 0, \quad x \in (\partial\Omega) \cap \partial U.$$

In the special case $K \supset \partial\Omega$, the problem is no longer mixed because we only have Dirichlet boundary conditions. We won't need the Neumann boundary equations in strong form, only the weak, integrated form. On the other hand, we will use continuity of the solutions up to the boundary. In fact, we will obtain C^α regularity.

To ensure the Hölder regularity of solutions we make an additional assumption on the compact set $K \subset M$. We will say that K has a *clean interface* with $\partial\Omega$ if at every point $x_0 \in (\partial K) \cap \partial(\Omega \setminus K)$, $K \subset \bar{\Omega}$ is locally equivalent by a bi-Lipschitz mapping to a quadrant in a half space

$$Q = \{y \in \mathbb{R}^n : y_1 \geq 0, y_2 \geq 0\} \subset H = \{y \in \mathbb{R}^n : y_2 \geq 0\}.$$

More precisely, there are $r > 0$ and $\varepsilon > 0$ and a bi-Lipschitz mapping $F : B_r(x_0) \rightarrow F(B_r(x_0)) \subset \mathbb{R}^n$ such that $F(x_0) = 0$ and

$$(B_\varepsilon(0) \cap Q) \subset F(K \cap B_r(x_0)) \subset Q, \quad (B_\varepsilon(0) \cap H) \subset F(\bar{\Omega} \cap B_r(x_0)) \subset H.$$

We assume in addition that K has a bi-Lipschitz boundary. In other words, near points of $\Omega \cap \partial K$, K is locally bi-Lipschitz equivalent to a half space.

The constants in the estimates in our main theorems do not depend on any of these constants, r , ε or the bi-Lipschitz constants because continuity of solutions is only used in a qualitative way.

Proposition 2.3. *Suppose that V is nondegenerate, K is a compact subset of M with bi-Lipschitz boundary and has a clean interface with $\partial\Omega$. Let U be a connected component of $M \setminus K$. Then there is an orthonormal basis φ_j of $L^2(U, m dx)$ of eigenfunctions solving $L\varphi_j = \mu_j\varphi_j$ weakly on U , $\mu_j > 0$. After extending the functions φ_j from U to the rest of M by $\varphi_j = 0$ on $M \setminus U$, they satisfy $\varphi_j \in C^\alpha(M) \cap W^{1,2}(M)$ for some $\alpha > 0$.*

The proof of the existence of the complete orthonormal basis of eigenfunctions is the same as in the case of $K = \emptyset$, that is, the case of ψ_j above. See Proposition 5.1 for the proof C^α regularity. (At interior points the proof is similar to the case of $Lv = f$ above. The boundary regularity is proved using an even reflection.)

3. AGMON ESTIMATES

We will frequently write

$$\nabla_A = A^{1/2}\nabla$$

in which $A^{1/2} = A^{1/2}(x)$ is the positive definite square root of the matrix $A(x)$ and ∇ is a column vector. Thus, we have

$$\nabla_A\varphi \cdot \nabla_A\eta = (A\nabla\varphi) \cdot (\nabla\eta); \quad |\nabla_A\varphi|^2 = (A\nabla\varphi) \cdot \nabla\varphi.$$

Lemma 3.1. *Assume that f and u belong to $W^{1,2}(M)$, that V , f , and $1/u$ belong to $L^\infty(M)$, and that u satisfies $Lu = 1$ weakly on M . Then*

$$\int_M (|\nabla_A f|^2 + V f^2) m dx = \int_M \left(u^2 |\nabla_A(f/u)|^2 + \frac{1}{u} f^2 \right) m dx.$$

Proof. The function f^2/u belongs to $W^{1,2}(M)$, so we may use it as a test function in the weak form of $Lu = 1$ to obtain

$$\int_M [(\nabla_A u \cdot \nabla_A(f^2/u)) + V u(f^2/u)] m dx = \int_M (f^2/u) m dx.$$

Substituting the identity $\nabla_A u \cdot \nabla_A(f^2/u) = |\nabla_A f|^2 - u^2 |\nabla_A(f/u)|^2$ (from the product rule), this becomes

$$\int_M (|\nabla_A f|^2 - u^2 |\nabla_A(f/u)|^2 + V f^2) m dx = \int_M (f^2/u) m dx,$$

which, after moving a term from the left to the right, is the desired result. \square

Given the importance of Lemma 3.1 to this paper, we wish to elaborate on it and show how one can discover it. Recall that

$$Lf = -\frac{1}{m} \operatorname{div}(mA\nabla f) + Vf$$

in the weak sense. Define the operator \tilde{L} by

$$\tilde{L}g := \frac{1}{u} L(gu).$$

In other words, \tilde{L} is the conjugation of L by the operator multiplication by u . If the functions m and A are differentiable, then one can use equation $Lu = 1$ to compute that

$$\tilde{L}g = -\frac{1}{mu^2} \operatorname{div}(mu^2 A \nabla g) + \frac{1}{u}g.$$

Note that the operator \tilde{L} is of the same form as L but with a different density and potential. The key point is that the potential V in L has been replaced by the potential $1/u$ in \tilde{L} . Mechanisms of this type are familiar in the theory of second order differential equations. Conjugation of operators of the form $-\Delta + V$ using an auxiliary solution is a standard device leading to the generalized maximum principle (see Theorem 10, page 73 [PW]). A similar device appears even earlier in work of Jacobi on conjugate points and work of Sturm on oscillation of eigenfunctions. In all of these cases, the multipliers are eigenfunctions or closely related supersolutions rather than solutions to the equation $Lu = 1$.

Consider the space $L^2(M, m dx)$ with inner product $\langle \cdot, \cdot \rangle$. The operators L and $u^2 \tilde{L}$ are self adjoint in this inner product. Using the formula for \tilde{L} above, one could derive the lower bound $\langle Lf, f \rangle \geq \langle (1/u)f, f \rangle$ formally by substituting $f = gu$:

$$\langle Lf, f \rangle = \langle u^2 \tilde{L}g, g \rangle \geq \langle u^2 (1/u)g, g \rangle = \langle (1/u)f, f \rangle.$$

Lemma 3.1 implies that the identity $\langle Lf, f \rangle = \langle u^2 \tilde{L}g, g \rangle$ is valid in weak form. Indeed, it says that

$$\langle Lf, f \rangle = \int_M (|\nabla_A f|^2 + V f^2) m dx = \int_M \left[u^2 |\nabla_A (f/u)|^2 + \frac{1}{u} f^2 \right] m dx,$$

and so, since $g = f/u$,

$$\langle Lf, f \rangle = \int_M u^2 \left[|\nabla_A g|^2 + \frac{1}{u} g^2 \right] m dx = \langle u^2 \tilde{L}g, g \rangle.$$

Although conjugation and the calculation of \tilde{L} leads to our identity, the weak form has considerable advantages. It is easier to check the weak formula than the differential formula for \tilde{L} because it only involves first derivatives. Moreover, because we only differentiated once and didn't integrate by parts, our proof of Lemma 3.1 was not only shorter but also more general in that it applied to bounded measurable m and A .

We will now derive estimates of Agmon type from Lemma 3.1.

Lemma 3.2. *Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on a compact subset K of M and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Let u be as in Lemma 3.1 and let g be a Lipschitz function on M . Then*

$$(3.1) \quad \int_M \left[u^2 |\nabla_A (g\varphi/u)|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right] m dx = \int_M |\nabla_A g|^2 \varphi^2 m dx.$$

Furthermore, setting $g = \chi e^h$ with h and χ Lipschitz functions on M , we have

$$(3.2) \quad \int_M u^2 \left| \nabla_A \left(\frac{\chi e^h \varphi}{u} \right) \right|^2 m dx + \int_M \left(\frac{1}{u} - \mu - |\nabla_A h|^2 \right) (\chi e^h \varphi)^2 m dx \\ = \int_M (|\chi \nabla_A h + \nabla_A \chi|^2 - |\chi \nabla_A h|^2) (e^h \varphi)^2 m dx.$$

Proof. Since $g^2 \varphi \in W^{1,2}(M)$ and $g^2 \varphi = 0$ on K , it can be used as a test function for the equation $L\varphi = \mu\varphi$, yielding

$$(3.3) \quad \int_M (V - \mu) g^2 \varphi^2 m dx = - \int_M \nabla_A \varphi \cdot \nabla_A (g^2 \varphi) m dx.$$

Substituting $f = g\varphi$ in Lemma 3.1, gives

$$\int_M [|\nabla_A(g\varphi)|^2 + (V - \mu)g^2\varphi^2] m dx \\ = \int_M \left[u^2 |\nabla_A(g\varphi/u)|^2 + \left(\frac{1}{u} - \mu \right) g^2 \varphi^2 \right] m dx.$$

On the other hand, (3.3) implies that

$$\int_M [|\nabla_A(g\varphi)|^2 + (V - \mu)g^2\varphi^2] m dx \\ = \int_M [|\nabla_A(g\varphi)|^2 - \nabla_A \varphi \cdot \nabla_A (g^2 \varphi)] m dx = \int_M \varphi^2 |\nabla_A g|^2 m dx.$$

This proves (3.1). The second formula, (3.2), follows from the first, by setting $g = \chi e^h$, and using the formula

$$|\nabla_A g|^2 = |\nabla_A(\chi e^h)|^2 = (\chi e^h)^2 |\nabla_A h|^2 + (|\chi \nabla_A h + \nabla_A \chi|^2 - |\chi \nabla_A h|^2) e^{2h}.$$

□

Let w be a nonnegative, continuous function on M . Assume the elliptic matrix A is continuous on M . Denote the entries of $B = A^{-1}$ by $b_{ij}(x)$. We define the distance $\rho(x, y)$ on M for the degenerate Riemannian metric $ds^2 = w(x) \sum b_{ij} dx_i dx_j$ by

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i,j=1}^n b_{ij}(\gamma(t)) \dot{\gamma}_i(t) \dot{\gamma}_j(t) \right)^{1/2} dt,$$

where the infimum is taken over all absolutely continuous paths $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. (Note that the distance between points in a connected component of the set $\{w = 0\}$ is zero.)

With these notations, we have the following lemma.

Lemma 3.3 ([A, Theorem 4, p. 18]). *If h is real-valued and $|h(x) - h(y)| \leq \rho(x, y)$ for all $x, y \in M$, then h is a Lipschitz function, and*

$$|\nabla_A h(x)|^2 \leq w(x) \quad \text{for all } x \in M.$$

In particular, this holds when

$$h(x) = \inf_{y \in E} \rho(x, y),$$

for any nonempty set $E \subset M$.

The lemma is stated in [A] for w strictly positive. Considering the case $w(x) + \epsilon$ and taking the limit as $\epsilon \searrow 0$ gives the result for non-negative w .

Recall that V is a measurable function on M such that $0 \leq V(x) \leq \bar{V}$, and V is nonzero on a set of positive measure and u is the unique weak solution to $Lu = 1$ on M , the landscape function.

Fix $\mu \geq 0$, and set

$$w_\mu(x) = \left(\frac{1}{u(x)} - \mu \right)_+ = \max \left(\frac{1}{u(x)} - \mu, 0 \right).$$

With our additional assumption that the elliptic matrix A has continuous coefficients on M , we can define $\rho_\mu(x, y)$ as the Agmon distance associated to the weight $w_\mu(x)$. For any $E \subset M$, denote

$$\rho_\mu(x, E) = \inf_{y \in E} \rho_\mu(x, y).$$

Theorem 3.4. *Let $0 \leq \mu \leq \nu \leq \bar{V}$ be constants. With u the landscape function as above, denote*

$$E(\nu) = \left\{ x \in M : \frac{1}{u(x)} \leq \nu \right\}.$$

Let K be a compact subset of M . Denote

$$h(x) = \rho_\mu(x, E(\nu) \setminus K), \quad x \in M,$$

and

$$\chi(x) = \begin{cases} h(x), & h(x) < 1, \\ 1, & h(x) \geq 1. \end{cases}$$

Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on K , and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Then for $0 < \alpha < 1$,

(3.4)

$$\begin{aligned} \int_M u^2 \left| \nabla_A \left(\frac{\chi e^{\alpha h} \varphi}{u} \right) \right|^2 m \, dx + (1 - \alpha^2) \int_M \left(\frac{1}{u} - \mu \right)_+ \left(\chi e^{\alpha h} \varphi \right)^2 m \, dx \\ \leq (1 + 2\alpha) e^{2\alpha(\bar{V} - \mu)} \int_{\{0 < h < 1\}} \varphi^2 m \, dx. \end{aligned}$$

Furthermore, if $\nu = \mu + \delta$, $\delta > 0$, we have

$$(3.5) \quad \int_{h \geq 1} e^{2\alpha h} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) m \, dx \leq \left(450 + \frac{130\bar{V}}{(1 - \alpha)\delta} \right) \bar{V} \int_M \varphi^2 m \, dx.$$

Proof. Using (3.2) with αh in place of h , the first term on the left side is the same as in (3.4). Since $\chi = 0$ on $E_\mu \setminus K$ and $\varphi = 0$ on K , we have $\chi\varphi = 0$ on E_μ . Moreover, by Lemma 3.3 $|\nabla_A h|^2 \leq w_\mu(x)$. Thus,

$$\begin{aligned} & \int_M \left(\frac{1}{u} - \mu - \alpha^2 |\nabla_A h|^2 \right) (\chi e^{\alpha h} \varphi)^2 m dx \\ &= \int_{M \setminus E_\mu} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla_A h|^2 \right) (\chi e^{\alpha h} \varphi)^2 m dx \\ &\geq (1 - \alpha^2) \int_{M \setminus E_\mu} \left(\frac{1}{u} - \mu \right)_+ (\chi e^{\alpha h} \varphi)^2 m dx \\ &= (1 - \alpha^2) \int_M \left(\frac{1}{u} - \mu \right)_+ (\chi e^{\alpha h} \varphi)^2 m dx. \end{aligned}$$

The right side integrand of (3.2) is zero almost everywhere on the set $\nabla_A \chi = 0$, so we may restrict the integral to the set $\{0 < h < 1\}$. There we have $\chi \equiv h$, so

$|\chi \alpha \nabla_A h + \nabla_A \chi|^2 - |\chi \alpha \nabla_A h|^2 = [(\chi \alpha + 1)^2 - \chi^2 \alpha^2] |\nabla_A h|^2 \leq (2\alpha + 1) |\nabla_A h|^2$. Finally, $|\nabla_A h|^2 \leq w_\mu(x) \leq \bar{V} - \mu$, by Lemma 3.3 and Proposition 2.2. This concludes the proof of (3.4).

It remains to prove (3.5). For convenience, normalize φ so that its $L^2(M, m dx)$ norm is 1:

$$\|\varphi\|^2 := \int_M \varphi^2 m dx = 1.$$

Let $f = \chi e^{\alpha h} \varphi$. Since $f = 0$ on $E(\nu)$, $(1/u - \mu) \geq \delta$ on $M \setminus E(\nu)$, and $\mu \geq 0$, (3.4) implies

$$(3.6) \quad \int_M u^2 |\nabla_A(f/u)|^2 m dx + (1 - \alpha^2) \delta \int_M f^2 m dx \leq (1 + 2\alpha) e^{2\alpha} \bar{V}.$$

Since ∇f and ∇u belong to $L^2(M)$, and $1/u$ and f belong to $L^\infty(M)$, f^2/u is a permissible test function. Thus, using $Lu = 1$, $1/u(x) \leq \bar{V}$, $V(x) \geq 0$, and (3.6), we have

$$(3.7) \quad \begin{aligned} & \int_M \nabla_A u \cdot \nabla_A(f^2/u) m dx = \int_M (1 - Vu)(f^2/u) m dx \\ & \leq \bar{V} \int_M f^2 m dx \leq \frac{(1 + 2\alpha) e^{2\alpha}}{(1 - \alpha^2) \delta} \bar{V}^2 \leq \frac{3e^2}{2(1 - \alpha) \delta} \bar{V}^2. \end{aligned}$$

Next,

$$\begin{aligned} & \int_M |\nabla_A u|^2 (f/u)^2 m dx = - \int_M 2(f/u)(\nabla_A u) \cdot (u \nabla_A(f/u)) m dx \\ & \quad + \int_M \nabla_A u \cdot \nabla_A(f^2/u) m dx \\ & \leq \int_M \left[\frac{1}{2} (f/u)^2 |\nabla_A u|^2 + 2u^2 |\nabla_A(f/u)|^2 + \nabla_A u \cdot \nabla_A(f^2/u) \right] m dx. \end{aligned}$$

Hence, after subtracting the term with factor 1/2 and multiplying by 2,

$$\begin{aligned} \int_M |\nabla_A u|^2 (f/u)^2 m dx &\leq \int_M [4u^2 |\nabla_A(f/u)|^2 + 2\nabla_A u \cdot \nabla_A(f^2/u)] m dx \\ &\leq 4(1+2\alpha)e^{2\alpha}\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \\ &\leq 12e^2\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}. \end{aligned}$$

It follows that

$$\begin{aligned} (3.8) \quad \int_M |\nabla_A f|^2 m dx &= \int_M |u\nabla_A(f/u) + (f/u)\nabla_A u|^2 m dx \\ &\leq 2 \int_M u^2 |\nabla_A(f/u)|^2 m dx + 2 \int_M |\nabla_A u|^2 (f/u)^2 m dx \\ &\leq 2(1+2\alpha)e^{2\alpha}\bar{V} + 2 \left[12e^2\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \right] m dx \\ &\leq 30e^2\bar{V} + 6e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}. \end{aligned}$$

Finally, since $e^{\alpha h}\varphi = f$ on $\{h \geq 1\}$, and $|\nabla_A h|^2 \leq \bar{V}$, we have (by (3.7) and (3.8) in particular)

$$\begin{aligned} (3.9) \quad \int_{\{h \geq 1\}} e^{2\alpha h} |\nabla_A \varphi|^2 m dx &= \int_{\{h \geq 1\}} |\nabla_A(e^{\alpha h}\varphi) - \alpha(\nabla_A h)e^{\alpha h}\varphi|^2 m dx \\ &\leq 2 \int_{\{h \geq 1\}} |\nabla_A(e^{\alpha h}\varphi)|^2 m dx + 2 \int_{\{h \geq 1\}} \alpha^2 |\nabla_A h|^2 (e^{\alpha h}\varphi)^2 m dx \\ &\leq 2 \int_{\{h \geq 1\}} |\nabla_A f|^2 m dx + 2\bar{V} \int_{\{h \geq 1\}} f^2 m dx \\ &\leq 60e^2\bar{V} + 12e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}. \end{aligned}$$

Thus, by (3.7) again,

$$\begin{aligned} \int_{\{h \geq 1\}} e^{2\alpha h} (|\nabla_A \varphi|^2 + \bar{V}\varphi^2) m dx &\leq 60e^2\bar{V} + 15e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} + \frac{3}{2}e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \\ &\leq \left(450 + \frac{130\bar{V}}{(1-\alpha)\delta} \right) \bar{V}. \end{aligned}$$

□

Theorem 3.4 displays the dependence of the constant as $\alpha \rightarrow 1$. We state next a variant for $\alpha = 1/2$ in the form we will use below.

Corollary 3.5. *Let $0 < \mu \leq \bar{\mu}$ and $0 < \delta \leq \bar{V}/10$ be constants. Suppose that $\bar{\mu} + \delta \leq \bar{V}$. Let K be a compact subset of M , and set*

$$h_K(x) = \bar{\rho}(x, E(\bar{\mu} + \delta) \setminus K), \quad x \in M,$$

with $\bar{\rho} = \rho_{\bar{\mu}}$ the Agmon metric associated to the weight $\bar{w}(x) = (1/u(x) - \bar{\mu})_+$. Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on K , and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Then

$$(3.10) \quad \int_{h_K \geq 1} e^{h_K} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) m dx \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V} \int_M \varphi^2 m dx.$$

In particular, in the case $K = \emptyset$, the corollary says that for eigenfunctions ψ satisfying $L\psi = \lambda\psi$ weakly on all of M for which $\lambda \leq \bar{\mu}$, we have

$$(3.11) \quad \int_{h \geq 1} e^h (|\nabla_A \psi|^2 + \bar{V} \psi^2) m dx \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V} \int_M \psi^2 m dx.$$

with

$$h(x) = \bar{\rho}(x, E(\bar{\mu} + \delta)), \quad x \in M.$$

Proof. Corollary 3.5 is not, strictly speaking, a corollary of Theorem 3.4, but rather the specialization of the inequalities in the proof to the case $\alpha = 1/2$. Note also the theorem is proved for $\mu = \bar{\mu}$, but the corollary is also valid for any larger value of $\bar{\mu}$. This because increasing $\bar{\mu}$ gives rise to a weaker conclusion: it decreases h_K .

Rather than repeat the proof, we indicate briefly the arithmetic that ensues from setting $\alpha = 1/2$ in the proof of Theorem 3.4. With $f = \chi e^{h_K/2} \varphi$ and the normalization $\|\varphi\| = 1$, we have

$$\int_{\{h_K \geq 1\}} e^{h_K} \bar{V} \varphi^2 m dx \leq \bar{V} \int_M f^2 m dx \leq \frac{8e \bar{V}^2}{3 \delta},$$

as in the second line of (3.7),

$$\int_M |\nabla_A f|^2 m dx \leq \left(20 + \frac{3\bar{V}}{\delta} \right) e \bar{V},$$

by the proof of (3.8), and (as for (3.9))

$$\int_{\{h_K \geq 1\}} e^{h_K} |\nabla_A \varphi|^2 m dx \leq \left(40 + \frac{34\bar{V}}{3\delta} \right) e \bar{V}.$$

Therefore, again with the normalization $\|\varphi\| = 1$,

$$\int_{\{h_K \geq 1\}} e^{h_K} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) m dx \leq \left(40 + 14 \frac{\bar{V}}{\delta} \right) e \bar{V} \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V},$$

where we have used $\delta \leq \bar{V}/10$ to obtain the last inequality. \square

4. LOCALIZED APPROXIMATE EIGENFUNCTIONS

We have already proved a theorem about exponential decay of the eigenfunctions ψ . We will now show, roughly speaking, that if the landscape function predicts localization, then an eigenfunction with eigenvalue λ is localized in the components of $\{1/u \leq \lambda\}$ where an appropriate localized problem has an eigenvalue in the range $\lambda \pm \delta$.

Let $\bar{\mu}$ and δ be as in Corollary 3.5. Consider any finite decomposition of the sublevel set $E(\bar{\mu} + \delta)$ into subsets:

$$E(\bar{\mu} + \delta) = \{x \in M : \frac{1}{u(x)} \leq \bar{\mu} + \delta\} = \bigcup_{\ell=1}^R E_\ell.$$

We regard the sets E_ℓ as potential wells. It is easiest to visualize E_ℓ as the (closed) connected components of $E(\bar{\mu} + \delta)$. In practice, such connected wells often yield the optimal result. But there is no requirement that E_ℓ be connected. Rather each E_ℓ should be chosen to consist of a collection of “nearby” wells. It is occasionally useful to merge nearby wells because what is important is to choose the sets E_ℓ so as to have a large separation between them, where the separation S is defined by

$$S = \inf \{\bar{\rho}(x, y) : x \in E_\ell, y \in E_{\ell'}, \ell \neq \ell'\},$$

i.e., the smallest Agmon distance between wells. Here, as before, $\bar{\rho} = \rho_{\bar{w}}$ denotes the Agmon metric associated to the weight $\bar{w}(x) = (1/u(x) - \bar{\mu})_+$. Whether or not a decomposition into small, well-separated wells exists depends on the level set structure of $1/u(x)$ and the size of $\bar{\mu} + \delta$.

Let $S_1 < S$ (as near to S as we like). We claim that there is a compact set $K_\ell \subset M = \bar{\Omega}$ satisfying the hypothesis of Proposition 2.3 and such that

$$(4.1) \quad \{x \in M : \bar{\rho}(x, E_\ell) \geq S/2\} \subset K_\ell \subset \{x \in M : \bar{\rho}(x, E_\ell) > \frac{S_1}{2}\}.$$

In fact, as we will show in Lemma 5.2, for any compact $K \subset M$ and any neighborhood $U \supset K$ (that is, U is relatively open in M) there is an intermediate set $K \subset K' \subset U$ such that K' has a bi-Lipschitz boundary and a clean interface with $\partial\Omega$. This lemma is where the global hypothesis on $\partial\Omega$ is used. We will propose a simpler type of intermediate set K_ℓ in Section 6.

Define Ω_ℓ as the connected component of $M \setminus K_\ell$ containing E_ℓ . Because the sets E_ℓ are at least distance S apart, the sets Ω_ℓ are disjoint.

Denote by $W_0^{1,2}(\Omega_\ell)$ the closure in $W^{1,2}(M)$ norm of the space of smooth functions that are compactly supported on Ω_ℓ . Note that these functions can be extended by zero on $M \setminus \Omega_\ell$ and regarded as belonging to $W^{1,2}(M)$. But the notation is slightly misleading, because Ω_ℓ is not necessarily open, and may contain parts of ∂M that do not lie in K_ℓ . On those parts, functions of $W_0^{1,2}(\Omega_\ell)$ do not need to vanish. In other words, our definition of $W_0^{1,2}(\Omega_\ell)$ includes a Dirichlet condition on $K_\ell \cap \partial\Omega_\ell$ only.

The operator L is self-adjoint with our mixture of Dirichlet and Neumann conditions, and for each ℓ there a complete system of orthonormal

eigenfunctions $\varphi_{\ell,j} \in W_0^{1,2}(\Omega_\ell)$ satisfying

$$\int_M [\nabla_A \varphi_{\ell,j} \cdot \nabla \zeta + V \varphi_{\ell,j} \zeta] m dx = \mu_{\ell,j} \int_M \varphi_{\ell,j} \zeta m dx$$

for all test functions ζ in $W_0^{1,2}(\Omega_\ell)$. We have Dirichlet conditions on $K_\ell \cap \partial\Omega_\ell$. If $\partial\Omega_\ell \cap \partial M$ is non-empty, then on that portion of the boundary, the weak equation is interpreted as a Neumann condition. But we will never have to use normal derivatives, only the weak equation. The purpose of inserting the somewhat nicer domain Ω_ℓ is so that the eigenfunctions $\varphi_{\ell,j}$ are continuous (in fact Hölder continuous) on M . We do this so that the integrals in the lemmas above are well defined. None of our inequalities with exponential weights depend on the Lipschitz constant of Ω_ℓ , just as they don't depend on the ellipticity constant or modulus of continuity of A .

Let ψ_j denote the complete system of orthonormal eigenfunctions of L on M with eigenvalues λ_j . Let $\Psi_{(a,b)}$ denote the orthogonal projection in $L^2(M, m dx)$ onto the span of eigenvectors ψ_j with eigenvalue $\lambda_j \in (a, b)$. Let $\Phi_{(a,b)}$ be the orthogonal projection onto the span of the eigenvectors $\varphi_{\ell,j}$ with eigenvalue $\mu_{\ell,j} \in (a, b)$. Thus the range of $\Phi_{(0,\infty)}$ is $L^2\left(\bigcup_{\ell} \Omega_\ell\right)$.

Theorem 4.1. *Let $0 < \delta \leq \bar{V}/10$. If φ is one of the $\varphi_{\ell,j}$ with eigenvalue $\mu = \mu_{\ell,j}$ and $\mu \leq \bar{\mu} - \delta$, and S is the Agmon distance separating wells, defined using $\bar{\mu} + \delta \leq \bar{V}$ as above, then*

$$\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)}\varphi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta}\right)^3 e^{-S/2},$$

where here and below, $\|\cdot\|$ denotes the norm in $L^2(M, m dx)$. If ψ is one of the ψ_j with eigenvalue $\lambda = \lambda_j \leq \bar{\mu} - \delta$, then

$$\|\psi - \Phi_{(\lambda-\delta, \lambda+\delta)}\psi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta}\right)^3 e^{-S/2}.$$

Here and in the remainder of the paper all eigenfunctions are normalized to have $L^2(m dx)$ norm 1.

Note that this theorem only has content if S is sufficiently large that

$$\left(\frac{\bar{V}}{\delta}\right)^3 \ll e^{S/2}.$$

Recall that we have the flexibility to choose the sets E_ℓ so as to merge nearby wells that are not sufficiently separated. It turns out that the partition into well-separated wells does occur with high probability for many classes of random potentials V . The separation is easier and easier to achieve as the threshold $\bar{\mu} + \delta$ decreases.

Proof. Consider φ such that $L\varphi = \mu\varphi$ in the weak sense on Ω_ℓ . Let

$$\eta(x) = f(\bar{\rho}(x, E_\ell))$$

be defined by

$$f(t) = \begin{cases} 1, & t \leq \frac{S_1}{2} - 1 \\ \frac{S_1}{2} - t, & \frac{S_1}{2} - 1 \leq t \leq \frac{S_1}{2} \\ 0, & \frac{S_1}{2} \leq t \end{cases}$$

Let r be the distribution satisfying the equation

$$L(\eta\varphi) = \mu\eta\varphi + r$$

in the weak sense on M . In other words, r is defined by

$$r(\zeta) := \int_M [\nabla_A(\eta\varphi) \cdot \nabla_A\zeta + (V - \mu)\eta\varphi\zeta] m dx$$

for all ζ smooth functions on M . Since $\eta\zeta$ is a suitable test function for $L\varphi = \mu\varphi$ in Ω_ℓ , we have

$$\int_M [\nabla_A(\varphi) \cdot \nabla_A(\eta\zeta) + (V - \mu)\eta\varphi\zeta] m dx = 0.$$

Subtracting this formula from the previous one for r , we find that

$$r(\zeta) = \int_M [\varphi\nabla_A\eta \cdot \nabla_A\zeta - \zeta\nabla_A\varphi \cdot \nabla_A\eta] m dx.$$

Observe that if $\nabla_A\eta(x) \neq 0$, then $\frac{S_1}{2} - 1 \leq \bar{\rho}(x, E_\ell) \leq \frac{S_1}{2}$. Furthermore, since the distance from E_ℓ to $E_{\ell'}$, $\ell' \neq \ell$, is greater than S_1 , $\bar{\rho}(x, E_{\ell'}) \geq S_1/2$. Thus, since $E(\bar{\mu} + \delta) = \bigcup_{\ell=1}^R E_\ell$, we have $\bar{\rho}(x, E(\bar{\mu} + \delta)) \geq S_1/2 - 1$. In particular, for any set K , $\nabla_A\eta(x) \neq 0$ implies

$$h_K(x) = \bar{\rho}(x, E_{\bar{\mu}+\delta} \setminus K) \geq \frac{S_1}{2} - 1.$$

We use this, (3.10) with $K = M \setminus \Omega_\ell$, and $|\nabla_A\eta|^2 \leq \bar{V}$ to obtain (recall the normalization $\|\varphi\| = 1$)

$$\begin{aligned} |r(\zeta)| &\leq (\sup |\nabla_A\eta|) \|\nabla_A\zeta\| \left(\int_{\{\nabla_A\eta \neq 0\}} \varphi^2 m dx \right)^{1/2} \\ &\quad + (\sup |\nabla_A\eta|) \|\zeta\| \left(\int_{\{\nabla_A\eta \neq 0\}} |\nabla_A\varphi|^2 m dx \right)^{1/2} \\ &\leq \|\nabla_A\zeta\| \left(\int_{\{\nabla_A\eta \neq 0\}} \bar{V}\varphi^2 m dx \right)^{1/2} + \bar{V}^{1/2} \|\zeta\| \left(\int_{\{\nabla_A\eta \neq 0\}} |\nabla_A\varphi|^2 m dx \right)^{1/2} \\ &\leq (\|\nabla_A\zeta\|^2 + \bar{V}\|\zeta\|^2)^{1/2} \left(\frac{18e^2\bar{V}^2}{\delta e^{S_1/2}} \right)^{1/2}. \end{aligned}$$

We will abbreviate this inequality by

$$(4.2) \quad r(\zeta)^2 \leq \varepsilon \bar{V} [\|\nabla_A\zeta\|^2 + \bar{V}\|\zeta\|^2], \quad \varepsilon := 18e^2 \frac{\bar{V}}{\delta} e^{-S_1/2}.$$

Since $V(x) \geq 0$,

$$\|\nabla_A \psi_j\|^2 + \bar{V} \|\psi_j\|^2 \leq \int_M [|\nabla_A \psi_j|^2 + (V + \bar{V})\psi_j^2] m dx = \lambda_j + \bar{V}.$$

Let J be any finite list of indices j such that $|\lambda_j - \mu| \geq \delta$ and let

$$\zeta = \sum_{j \in J} \gamma_j \psi_j$$

be any linear combination of the ψ_j . By density considerations, such a ζ is admissible. Then, since $V \geq 0$,

$$\|\nabla_A \zeta\|^2 + \bar{V} \|\zeta\|^2 \leq \int_M [|\nabla_A \zeta|^2 + (V + \bar{V})\zeta^2] m dx = \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Consequently, it follows from (4.2) that

$$r(\zeta)^2 \leq \varepsilon \bar{V} [\|\nabla_A \zeta\|^2 + \bar{V} \|\zeta\|^2] \leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Denote by

$$\beta_j = \int_M \eta \varphi \psi_j m dx = \langle \eta \varphi, \psi_j \rangle$$

the coefficients of $\eta \varphi$ in the basis. Because $(L - \lambda_j)\psi_j = 0$ in the weak sense,

$$\begin{aligned} r(\zeta) &= \sum_{j \in J} \gamma_j \int_M [\nabla_A \psi_j \nabla_A (\eta \varphi) + (V - \lambda_j) \psi_j \eta \varphi] m dx \\ &\quad + \int_M \gamma_j (\lambda_j - \mu) \psi_j \eta \varphi m dx \\ &= \sum_{j \in J} \gamma_j (\lambda_j - \mu) \beta_j. \end{aligned}$$

Thus,

$$\left| \sum_{j \in J} \gamma_j (\lambda_j - \mu) \beta_j \right|^2 = r(\zeta)^2 \leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Setting $\gamma_j = \beta_j (\lambda_j + \bar{V})^{-1/2} \text{sgn}(\lambda_j - \mu)$, we find that

$$\left| \sum_{j \in J} \frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}} \beta_j^2 \right|^2 \leq \varepsilon \bar{V} \sum_{j \in J} \beta_j^2.$$

Since $\lambda_j \geq 0$ and $|\lambda_j - \mu| \geq \delta$,

$$\frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}} \geq \frac{\delta}{\sqrt{\bar{V}}}.$$

Therefore,

$$\sum_{j \in J} \beta_j^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Since the set J is an arbitrary finite subset of j such that $|\lambda_j - \mu| \geq \delta$, we have

$$\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Next, it follows from (3.10) and $1 - \eta(x) = 0$ on $\{\bar{\rho}(x, E_\ell) \leq S_1/2 - 1\}$ that

$$\bar{V}\|(1 - \eta)\varphi\|^2 \leq \bar{V} \int_{h_K \geq \frac{S_1}{2} - 1} \varphi^2 m \, dx \leq \varepsilon \bar{V} \quad (K = M \setminus \Omega_\ell),$$

which, since the projection $I - \Psi_{(\mu-\delta, \mu+\delta)}$ has operator norm 1, implies that

$$\|(1 - \eta)\varphi - \Psi_{(\mu-\delta, \mu+\delta)}((1 - \eta)\varphi)\|^2 \leq \varepsilon.$$

Finally, adding the bounds for $\varphi = (1 - \eta)\varphi + \eta\varphi$ and using $\delta \leq \bar{V}/10$, we get

$$\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)}\varphi\|^2 \leq 2\varepsilon \frac{\bar{V}^2}{\delta^2} + 2\varepsilon < 300 \frac{\bar{V}^3}{\delta^3} e^{-S/2}.$$

This is the first claim of the theorem.

The second claim has a similar proof with the roles of φ and ψ reversed. We will sketch each step, but the reader will need to refer regularly to the previous proof. Let ψ be a normalized eigenfunction of L on M with eigenvalue $\lambda \leq \bar{\mu}$. We use the same cutoff functions

$$\eta_\ell(x) = f(\bar{\rho}(x, E_\ell)),$$

introducing the subscript ℓ since ℓ is no longer fixed. Then define

$$\tilde{\eta} = \sum_{\ell} \eta_\ell.$$

Note that $\tilde{\eta}\psi$ is compactly supported in the union of the Ω_ℓ , and the Ω_ℓ are disjoint. Define the distribution \tilde{r} by the equation

$$L(\tilde{\eta}\psi) = \lambda\tilde{\eta}\psi + \tilde{r}.$$

By similar reasoning to the proof of the first claim, using (3.11) we have the analogue of (4.2), that for all $\zeta \in W^{1,2}(M)$,

$$\tilde{r}(\zeta)^2 \leq \varepsilon \bar{V} [\|\nabla_A \zeta\|^2 + \bar{V}\|\zeta\|^2], \quad \varepsilon = 18e^2 \frac{\bar{V}}{\delta} e^{-S_1/2}.$$

Take any finite set \tilde{J} of indices (ℓ, j) and denote

$$\tilde{\zeta} = \sum_{(\ell, j) \in \tilde{J}} \gamma_{\ell j} \varphi_{\ell j}.$$

In the same way as before, we deduce

$$\tilde{r}(\tilde{\zeta})^2 \leq \varepsilon \bar{V} \sum_{\tilde{J}} (\mu_{\ell j} + \bar{V}) \gamma_{\ell j}^2.$$

Moreover, as before, if we define

$$\beta_{\ell j} = \int_M \tilde{\eta}\psi \varphi_{\ell j} m \, dx.$$

We claim that

$$\tilde{r}(\tilde{\zeta}) = \sum_{\tilde{J}} \gamma_{\ell j} (\mu_{\ell j} - \lambda) \beta_{\ell j}.$$

This last identity is the only place where the proof is slightly different. Observe that because $(L - \mu_{\ell j})\varphi_{\ell j} = 0$ in the weak sense on Ω_ℓ and $\eta_{\ell'}$ has support disjoint from $\overline{\Omega}_\ell$ for all $\ell' \neq \ell$,

$$\begin{aligned} & \int_M [\nabla_A(\tilde{\eta}\psi)\nabla_A\varphi_{\ell j} + (V - \mu_{\ell j})\tilde{\eta}\psi\varphi_{\ell j}] m \, dx \\ &= \int_{\Omega_\ell} [\nabla_A(\eta_\ell\psi)\nabla_A\varphi_{\ell j} + (V - \mu_{\ell j})\eta_\ell\psi\varphi_{\ell j}] m \, dx = 0. \end{aligned}$$

This is the only aspect of the proof of the formula for $\tilde{r}(\tilde{\zeta})$ that differs from the one for $r(\zeta)$ above.

Now suppose that for every $(\ell, j) \in \tilde{J}$, $|\mu_{\ell j} - \lambda| \geq \delta$. Then, setting

$$\gamma_{\ell j} = \beta_{\ell j} (\mu_{\ell j} + \overline{V})^{1/2} \operatorname{sgn}(\mu_{\ell j} - \lambda),$$

we obtain

$$\sum_{\tilde{J}} \beta_{\ell j}^2 \leq \varepsilon \frac{\overline{V}^2}{\delta^2}.$$

Since $\tilde{\eta}\psi$ is supported in the union $\bigcup_{\ell} \Omega_\ell$ and the $\varphi_{\ell j}$ are an orthonormal basis for L^2 on that set, and L is an arbitrary finite subset of indices such that $|\mu_{\ell j} - \lambda| \geq \delta$, we have

$$\|\tilde{\eta}\psi - \Phi_{(\lambda-\delta, \lambda+\delta)}(\tilde{\eta}\psi)\|^2 \leq \varepsilon \frac{\overline{V}^2}{\delta^2}.$$

Next, it follows from the fact that $(1 - \tilde{\eta}(x)) = 0$ on the set where $h(x) = \overline{\rho}(x, E(\overline{\mu} + \delta)) \leq S_1/2 - 1$ and (3.11) that

$$\overline{V} \|(1 - \tilde{\eta})\psi\|^2 \leq \overline{V} \int_{h \geq \frac{S_1}{2} - 1} \psi^2 m \, dx \leq \varepsilon \overline{V}.$$

The rest of the proof is similar. \square

Theorem 4.1 shows that when the landscape potential $1/u(x)$ defines wells that are separated by a large number S , then the eigenfunctions are located in these wells (with a single eigenfunction possibly occupying several wells). An easy consequence is the following corollary saying that the graphs of the two counting functions enumerating eigenvalues of L and eigenvalues localized to wells agree (modulo a shift $\pm\delta$) up to a number \overline{N} defined below.

Corollary 4.2. *Consider the counting functions*

$$N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}; \quad N_0(\mu) = \#\{\mu_{\ell, j} : \mu_{\ell, j} \leq \mu\}.$$

Recall that $\bar{\mu}$ and δ are used to specify S . Suppose that $\mu \leq \bar{\mu}$ and choose \bar{N} such that

$$300\bar{N} \left(\frac{\bar{V}}{\delta} \right)^3 < e^{S/2}.$$

Then

$$\min(\bar{N}, N_0(\mu - \delta)) \leq N(\mu) \quad \text{and} \quad \min(\bar{N}, N(\mu - \delta)) \leq N_0(\mu).$$

Proof. Let

$$m = \min(\bar{N}, N(\mu - \delta))$$

Consider the first m eigenvectors ψ_1, \dots, ψ_m of L on M . Then $m \leq N(\mu - \delta)$ implies $\lambda_j \leq \mu - \delta$, and therefore

$$\|\psi_j - \Phi_{(0,\mu)}\psi_j\|^2 \leq 300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-S/2}.$$

For any nonzero linear combination $\psi = \sum_{j=1}^m \alpha_j \psi_j$, we have

$$\begin{aligned} \|\psi - \Phi_{(0,\mu)}\psi\| &\leq \sum_j |\alpha_j| \|\psi_j - \Phi_{(0,\mu)}\psi_j\| \\ &\leq \left(300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-S/2} \right)^{1/2} \sum_j |\alpha_j| \\ &\leq \left(300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-S/2} \right)^{1/2} \|\psi\| m^{1/2} < \|\psi\|, \end{aligned}$$

by the Cauchy-Schwarz inequality and because $m \leq \bar{N}$. Denote by Q the span of the ψ_j , $j = 1, \dots, m$. The inequality implies the restriction of $\Phi_{(0,\mu)}$ to Q is injective and the dimension $N_0(\mu)$ of $\Phi_{(0,\mu)}(Q)$ is at least m . In other words, $N_0(\mu) \geq m$. The proof of the lower bound for $N(\mu)$ is similar. \square

5. MANIFOLDS AND APPROXIMATION

In this section we discuss two generalizations of the results of Section 2: the extension to manifolds and the removal of the continuity assumption on the coefficients of A . We also prove the boundary regularity for mixed data referred to in Section 2 and construct the intermediate sets K_ℓ with the clean interface property.

Let us first see how to replace \mathbb{R}^n with an ambient space \widehat{M} defined as a compact, connected C^1 manifold. Let V be a bounded measurable function satisfying $0 \leq V(x) \leq \bar{V}$ on \widehat{M} . Let A be a symmetric two-tensor and let m be a density on \widehat{M} . In a coordinate chart, x , A is represented locally by a symmetric matrix-valued function (which we shall still denote by A) and m is represented by a scalar function. Given a test function $\eta = \eta(x)$

compactly supported in the coordinate chart, and a function $\varphi = \varphi(x)$, we write

$$\langle A\nabla\varphi, \nabla\eta \rangle := \int (A\nabla\varphi) \cdot \nabla\eta m \, dx, \quad \langle \varphi, \eta \rangle := \int \varphi \eta m \, dx.$$

We extend these definitions to test functions on all of \widehat{M} by using a partition of unity. The covariance property that makes this definition independent of the choice of coordinate charts is that in a new coordinate system y with $x = x(y)$, the expression for the corresponding matrix $\widetilde{A}(y)$ and density $\widetilde{m}(y)$ is

$$\widetilde{A}(y) = B(y)^{-1} A(x(y)) (B(y)^{-1})^T, \quad \widetilde{m}(y) = |\det B| m(x(y)),$$

where B is the Jacobian matrix

$$B_{ij}(y) = \frac{\partial x_i}{\partial y_j}, \quad B = (B_{ij}).$$

For η supported in the intersection (in the x variable) of the two coordinate charts, denoting $\widetilde{\eta}(y) = \eta(x(y))$, $\widetilde{\varphi}(y) = \varphi(x(y))$, and $\widetilde{V}(y) = V(x(y))$, we have

$$\int [(A\nabla\varphi) \cdot \nabla\eta + V\varphi\eta] m \, dx = \int [(\widetilde{A}\nabla\widetilde{\varphi}) \cdot \nabla\widetilde{\eta} + \widetilde{V}\widetilde{\varphi}\widetilde{\eta}] \widetilde{m} \, dy.$$

Thus we obtain globally defined quantities $\langle A\nabla\varphi, \nabla\eta \rangle$ and $\langle V\varphi, \eta \rangle$.

We will assume that in some family of coordinate charts covering all of \widehat{M} , A is represented by bounded measurable, uniformly elliptic matrices and that m is bounded above and below by positive constants. The constant of ellipticity and the constants bounding m from above and below depend on the coordinate charts. But since our estimates won't depend on these constants, this does not matter to us.

The domains Ω in the C^1 manifold \widehat{M} that we will consider are bi-Lipschitz images of smoothly bounded domains. Let N be a compact C^∞ manifold with C^∞ boundary and let $F : N \rightarrow \widehat{M}$ be a bi-Lipschitz mapping from N to its image $F(N)$. Set $M = F(N)$ and $\Omega = F(N^\circ)$, the image of the interior of N . Thus $F(N) = \overline{\Omega}$. The theorems of the preceding sections are valid on $M \subset \widehat{M}$ for continuous A with no essential changes in the proofs.

Denote the inner product associated to $L^2(M)$ with density m by $\langle \cdot, \cdot \rangle$. Let K be a compact subset of M and let $W_0^{1,2}(M \setminus K)$ denote the closure in $W^{1,2}$ norm of the set of functions in $C^1(M)$ that vanish on K . For $\varphi \in W_0^{1,2}(M \setminus K)$ and $f \in L^2(M \setminus K)$, the weak equation $L\varphi = f$ on $M \setminus K$ is defined by

$$\langle A\nabla\varphi, \nabla\eta \rangle + \langle V\varphi, \eta \rangle = \langle f, \eta \rangle$$

for every $\eta \in W_0^{1,2}(M \setminus K)$.

We will now prove Hölder regularity of solutions up to the boundary for suitable K and f .

Proposition 5.1. *Suppose that K has a locally bi-Lipschitz boundary and has clean interface with ∂M , as defined above Proposition 2.3. There is $\alpha > 0$ such that if $f \in L^\infty(M)$ and $\varphi \in W^{1,2}(M)$, with $\varphi = 0$ on K , solves $(L - \mu)\varphi = f$ in the weak sense on $M \setminus K$, then $\varphi \in C^\alpha(M)$.*

Proof. Without loss of generality, we can replace V by $V - \mu$ and assume the constant $\mu = 0$. As we have already observed, the interior Hölder regularity follows from the theorem of De Giorgi-Nash-Moser. We show here how the boundary regularity can be reduced to the interior case. At boundary Neumann boundary points this is accomplished by an even reflection and at Dirichlet boundary points by an odd reflection. We will carry out the argument in a neighborhood of the interface between Dirichlet and Neumann conditions, since this is the most complicated and covers the other cases implicitly.

It suffices to consider a single coordinate chart denoted here by y . Let

$$B_r = \{y \in \mathbb{R}^n : |y| < r\}, \quad Q = \{y \in \mathbb{R}^n : y_1 \geq 0, y_2 \geq 0\}.$$

We will consider the domain $B_1 \cap Q$ and impose Neumann conditions on the $y_1 = 0$ plane and Dirichlet boundary conditions on the $y_2 = 0$ plane. The Dirichlet condition is imposed by assuming that φ is in the closure in $W^{1,2}(B_1 \cap Q)$ of $C^1(\overline{B_1} \cap Q)$ functions that vanish on $\{y_2 = 0\}$. For $f \in L^\infty(B_1 \cap Q)$ we say φ solves $L\varphi = f$ weakly in the sense that

$$\int_{B_1 \cap Q} [(A\nabla\varphi) \cdot \nabla\eta + V\varphi\eta] m \, dy = \int_{B_1 \cap Q} f\eta m \, dy$$

for all $\eta \in C^1(\overline{B_1} \cap Q)$ that vanish on ∂B_1 and $y_2 = 0$. (The fact that η need not vanish on $y_1 = 0$ is what imposes the Neumann condition in the weak sense.) Here, as usual, A is a bounded measurable symmetric matrix, f , V and m bounded measurable functions defined in $B_1 \cap Q$. Moreover, A is elliptic (see (2.1)) and $1/C \leq m(y) \leq C$.

To complete the proof, we will extend the L , φ and f to B_1 and show that the extended equation is valid on B_1 . Thus by the theorem of De Giorgi-Nash-Moser, the extension of φ belongs to $C^\alpha(B_{1/2})$.

Let R_1 and R_2 be the reflections,

$$R_1(y_1, y_2, \dots, y_n) = (-y_1, y_2, \dots, y_n), \quad R_2(y) = (y_1, -y_2, y_3, \dots, y_n).$$

Extend A , m and V to B_1 so that $\tilde{A}(y) = A(y)$, $\tilde{m}(y) = m(y)$, $\tilde{V}(y) = V(y)$ for $y \in B_1 \cap Q$, and

$$\tilde{A}(y) = R_j \tilde{A}(R_j y) R_j, \quad \tilde{m}(y) = \tilde{m}(R_j y), \quad \tilde{V}(y) = \tilde{V}(R_j y), \quad j = 1, 2.$$

Note that this is just the appropriate covariance for the changes of variable R_j since $R_j = R_j^T = R_j^{-1}$. In this way, we extend the definition of L to an operator \tilde{L} on B_1 .

Next, extend φ and f using the appropriate parity. Define $\tilde{\varphi}$ and \tilde{f} on B_1 by $\tilde{\varphi}(y) = \varphi(y)$ and $\tilde{f}(y) = f(y)$ for all $y \in B_1 \cap Q$ and

$$\tilde{\varphi}(y) = \tilde{\varphi}(R_1 y) = -\tilde{\varphi}(R_2 y), \quad \tilde{f}(y) = \tilde{f}(R_1 y) = -\tilde{f}(R_2 y).$$

Let $\eta \in C^1(\overline{B_1})$ be such that $\eta(y) = 0$ on ∂B_1 . Denote

$$\eta_*(y) = \frac{1}{4}(\eta(y) + \eta(R_1 y) - \eta(R_2 y) - \eta(R_1 R_2 y)), \quad y \in B_1.$$

Observe that the $*$ operation symmetrizes η , whereas $\tilde{\varphi}$ and \tilde{f} are defined so that they have this symmetry already: $\tilde{\varphi}_* = \tilde{\varphi}$ and $\tilde{f}_* = \tilde{f}$.

Denote the inner products on $L^2(B_1)$ and $L^2(B_1 \cap Q)$ by $\langle \cdot, \cdot \rangle_{B_1}$ and $\langle \cdot, \cdot \rangle_Q$, respectively. Since $\tilde{\varphi} = \tilde{\varphi}_*$,

$$\langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta \rangle_{B_1} = \langle \tilde{A} \nabla \tilde{\varphi}_*, \nabla \eta \rangle_{B_1} = \langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta_* \rangle_{B_1}.$$

Furthermore, using the symmetries, the fact that $\eta_*(y) = 0$ on $\{y_2 = 0\}$, and the weak equation for φ on $B_1 \cap Q$,

$$\begin{aligned} \langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta_* \rangle_{B_1} &= 4 \langle A \nabla \varphi, \nabla \eta_* \rangle_Q \\ &= 4 \langle f, \eta_* \rangle_Q - 4 \langle V \varphi, \eta_* \rangle_Q = \langle \tilde{f}, \eta \rangle_{B_1} - \langle \tilde{V} \tilde{\varphi}, \eta \rangle_{B_1}. \end{aligned}$$

Combining these two equations,

$$\langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta \rangle_{B_1} + \langle \tilde{V} \tilde{\varphi}, \eta \rangle_{B_1} = \langle \tilde{f}, \eta \rangle_{B_1}.$$

In other words, $\tilde{L} \tilde{\varphi} = \tilde{f}$ weakly on B_1 as desired. \square

Lemma 5.2. *Let K be a compact subset of M . Let U be a (relatively) open set in M such that $K \subset U$. Then there is a compact set K' , such that $K \subset K' \subset U$, K' has a locally bi-Lipschitz boundary in the interior of M and has a clean interface with ∂M .*

Proof. By the global bi-Lipschitz mapping $F : N \rightarrow M$, the problem reduces to the analogous one for $F^{-1}(K)$ on N . Thus for the purposes of this proof, we can assume without loss of generality that M is a C^∞ manifold with C^∞ boundary.

We begin with the case in which the boundary of M is empty. By a lemma of Whitney, there is a function $f \in C^\infty(M \setminus K)$ that is comparable to the distance to K . (This function is constructed using a partition of unity subordinate to an approximate Whitney decomposition of $M \setminus K$.) By a theorem of Morse, the critical values of f are a set of Lebesgue measure 0 in \mathbb{R} . For any non-critical value $\varepsilon > 0$, we have $K' = \{x \in M : f(x) \leq \varepsilon\}$ is smoothly bounded. Taking ε sufficiently small, we have $K \subset K' \subset U$.

We will carry out an analogous construction in the case $K \cap \partial M \neq \emptyset$ using reflection. Note first that one can embed M as a compact subset of an open manifold \tilde{M} so that ∂M is a smoothly embedded submanifold of \tilde{M} . Moreover, there is an open neighborhood U of ∂M in \tilde{M} and a

smooth reflection function across M , that is a function $g : U \rightarrow U$ satisfying $g(g(x)) = x$ for all $x \in U$, $g(x) = x$ for all $x \in \partial M$ and

$$g(U \cap M^\circ) = U \cap (\widetilde{M} \setminus M).$$

Define a distance $d(x, y)$ on M . Given a compact set $K \subset M$, set

$$K_\delta = \{x \in K : d(x, \partial M) \leq \delta\}; \quad K^* = K \cup g(K_\delta).$$

(Note that for $\delta > 0$ sufficiently small $K_\delta \subset U$.) By Whitney's lemma there is f that is C^∞ on the complement of K^* in $M \cup U$, comparable to the distance to K^* , and 0 on K^* . For every $x \in U$, define a symmetrized distance by $f_0(x) = f(x) + f(g(x))$. For x close to ∂M , f_0 is also comparable to the distance to K^* . Using a partition of unity, we can find a function f_1 equal to f_0 very near ∂M and equal to f farther from ∂M and comparable to the distance to K^* in a small neighborhood of M . Consider a noncritical value ε of f_1 . Let $x \in \partial M$ and suppose that $f_1(x) = \varepsilon$. Choose local coordinates y so that $y(x) = 0$ and ∂M is locally in the plane $y_1 = 0$. Writing the reflection $g = g(y)$ in the y coordinates we see that $g'(0)$ is the diagonal reflection matrix with 11 entry -1 (and $+1$ in all other diagonal entries). Taking the derivative with respect to y_1 of the relation $f_1(g(y)) = f_1(y)$ at $y = 0$, we find that $(\partial/\partial y_1)f_1(0) = 0$. Since $\nabla f_1(0) \neq 0$, we conclude that the gradient of f_1 is nonzero in the tangential direction in ∂M . In other words, the level surface $f_1 = \varepsilon$ is smooth and perpendicular in the y coordinates to ∂M at x . In particular, we have clean interface at this point. \square

The last difficulty that we wish to address is that the Agmon length of paths is not defined for discontinuous A . Suppose that A is bounded and measurable (and symmetric and uniformly elliptic as in (2.1)). Using convolution on coordinate charts and a partition of unity, we find a sequence of A^ε of continuous uniformly elliptic two-tensors such that A^ε tends pointwise to A as $\varepsilon \rightarrow 0$. Denote by L and L_ε the operators on M corresponding formally in local coordinates to $-(1/m)\operatorname{div}(mA\nabla) + V$ and $-(1/m)\operatorname{div}(mA^\varepsilon\nabla) + V$.

Proposition 5.3. *Let λ_ε be a bounded sequence, and suppose that $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$ in the weak sense on M , and normalize the eigenfunctions by $\|\psi_\varepsilon\| = 1$ in $L^2(M)$. Then there is a subsequence $\varepsilon_j \rightarrow 0$ such that*

- a) ψ_{ε_j} has a limit ψ in $W^{1,2}(M)$ norm and in $C^\alpha(M)$ norm for some $\alpha > 0$.
- b) λ_{ε_j} has a limit λ and $L\psi = \lambda\psi$ in the weak sense on M .

Proof. By the nondegeneracy of V , the sequence ψ_ε is uniformly bounded in $W^{1,2}(M)$ norm. Moreover by de Giorgi-Nash-Moser regularity the sequence is bounded in $C^\beta(M)$ norm for some $\beta > 0$. Note that β can be chosen independently of ε because ellipticity constants of A^ε are uniformly controlled. By the compactness of $C^\beta(M)$ in $C^\alpha(M)$ for $\alpha < \beta$ and the weak compactness of the unit ball of $W^{1,2}(M)$, there is a subsequence $\varepsilon_j \rightarrow 0$ such that ψ_{ε_j} converges in $C^\alpha(M)$ norm to a function $\psi \in C^\alpha(M) \cap W^{1,2}(M)$.

Moreover, $\nabla\psi_{\varepsilon_j} \rightarrow \nabla\psi$ weakly in $L^2(M)$ and $\lambda_{\varepsilon_j} \rightarrow \lambda$ as $j \rightarrow \infty$. Hence, taking the weak limit in the equation $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$, we obtain, $L\psi = \lambda\psi$.

It remains to show that $\nabla\psi_{\varepsilon_j}$ tends to $\nabla\psi$ in $L^2(M)$ norm. Indeed, by the dominated convergence theorem,

$$(5.1) \quad \|(A^{\varepsilon_j} - A)\nabla\psi\| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

From now on, we will omit the subscript j from ε with the understanding that we have passed to a subsequence of the A^ε and the ψ_ε . It follows that, along this subsequence,

$$\langle (A - A^\varepsilon)\nabla\psi, \nabla\psi \rangle \rightarrow 0 \quad \text{and} \quad \langle A^\varepsilon\nabla\psi, \nabla\psi \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle.$$

Furthermore, since $\|\nabla\psi_\varepsilon\|$ is uniformly bounded and by (5.1),

$$\langle (A - A^\varepsilon)\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow 0.$$

This combined with the weak limit $\langle A\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle$ yields

$$\langle A^\varepsilon\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle.$$

Using the identity $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$, we write

$$\langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle = \lambda_\varepsilon - \langle V\psi_\varepsilon, \psi_\varepsilon \rangle \rightarrow \lambda - \langle V\psi, \psi \rangle.$$

Finally,

$$\langle A^\varepsilon\nabla(\psi_\varepsilon - \psi), \nabla(\psi_\varepsilon - \psi) \rangle = \langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle - 2\langle A^\varepsilon\nabla\psi, \nabla\psi_\varepsilon \rangle + \langle A^\varepsilon\nabla\psi, \nabla\psi \rangle.$$

The first term of this last expression, $\langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle \rightarrow \lambda - \langle V\psi, \psi \rangle$. The second term tends to $-2\langle A\nabla\psi, \nabla\psi \rangle$ and the third term to $\langle A\nabla\psi, \nabla\psi \rangle$. But $L\psi = \lambda\psi$ implies $\langle A\nabla\psi, \nabla\psi \rangle = \lambda - \langle V\psi, \psi \rangle$. Thus

$$\langle A^\varepsilon\nabla(\psi_\varepsilon - \psi), \nabla(\psi_\varepsilon - \psi) \rangle \rightarrow 0$$

along the subsequence and $\nabla\psi_\varepsilon$ tends in $L^2(M)$ norm to $\nabla\psi$. \square

Let A have bounded measurable coefficients and let A^ε be a continuous approximation as above. Then the compactness argument in the proposition also shows that the landscape function u_ε tends uniformly to the landscape function u along a suitable subsequence. Because the Agmon distance functions are uniformly Lipschitz, at the expense of a further subsequence, one can ensure that this distance also converges uniformly. Notice that different sequences could, in principle, yield different limiting Agmon distances. For any of the limits we can now deduce estimates analogous to the ones in the previous sections.

We illustrate with (3.11) and discuss the subsequent theorems later. Fix μ and let W_μ be the subspace of $L^2(M)$ spanned by the eigenfunctions of L with eigenvalue $\leq \mu$, and let N be the dimension of W_μ . Denote

$$\mu_\varepsilon = \sup_{\psi \in W_\mu} \frac{\langle L_\varepsilon\psi, \psi \rangle}{\langle \psi, \psi \rangle}.$$

Let ψ_j^ε , $j = 1, \dots, N$ be the first N eigenfunctions of L_ε , and let λ_j^ε be the corresponding eigenvalues. It follows from the min/max principle and the fact that W_μ has dimension N that $\lambda_j^\varepsilon \leq \mu_\varepsilon$, $j \leq N$.

We claim that

$$(5.2) \quad \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon \leq \mu.$$

In fact, if ψ_j satisfying $L\psi_j = \lambda_j\psi_j$, $j = 1, \dots, N$, is an orthonormal basis of W_μ , then by the dominated convergence theorem, for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$|\langle L_\varepsilon \psi_j, \psi_k \rangle - \delta_{jk} \lambda_j| \leq \delta.$$

Representing ψ as a linear combination of the ψ_j , we deduce from $\lambda_j \leq \mu$ that $\mu_\varepsilon \leq \mu + N^2\delta$. Hence (5.2) holds.

By Proposition 5.3, for a suitable subsequence of values of ε the orthonormal basis ψ_j^ε , $j \leq N$, tends in $C^\alpha(M)$ and $W^{1,2}(M)$ norm to an orthonormal set of eigenfunctions of L with eigenvalues $\leq \mu$. Since W_μ has dimension N , this limiting set must be a basis for W_μ . Moreover, these eigenfunctions inherit the inequality (3.11).

There is a difference between this statement and the preceding one, applicable to continuous A . Here we only claim that there exists a basis of the eigenfunctions that satisfies (3.11). If an eigenvalue has multiplicity then the estimate may not apply to all linear combinations of the particular eigenbasis we obtain by taking limits. Thus, we have not ruled out the possibility that there has to be an extra factor of the multiplicity of the eigenspace in inequality (3.11). Similarly, in the comparisons with localized eigenfunctions in Theorem 4.1, we can only deduce that they are valid for *some* basis of eigenfunctions ψ_j and $\varphi_{\ell,j}$.

We leave open whether in the case of discontinuous A , it is possible to recover the full theorem for continuous coefficients for eigenfunctions with multiplicity. Another question that we are leaving open in the discontinuous case is whether the limiting Agmon distance is unique, that is, does not depend on the choice of the sequence A^ε . Even if the limit is not unique, there could be an optimal (largest) choice of h satisfying the Agmon bound $|\nabla_A h|^2 \leq w_\mu(x)$.

6. LOCALLY BI-LIPSCHITZ BOUNDARIES

So far, we have only proved our theorems for domains that are globally bi-Lipschitz equivalent to smooth domains. The place where this hypothesis is used is in the proof of C^α regularity up to the boundary in Dirichlet and mixed boundary value problems for $L - \mu$ for $\mu \in \mathbb{R}$. Regularity is a local property for elliptic equations, so it would be surprising if this global condition were necessary. We used it to surround level sets of the Agmon distance function by nearby smooth level sets that meet the fixed boundary $\partial\Omega$ transversally (what we called a clean interface). Then by means of

both odd and even reflections, we reduced our regularity problem to an interior problem and quoted a version of the De Giorgi–Nash–Moser interior regularity theorem (Theorem 8.24, [GT]).

In this section, we sketch a proof that we expect will work in general. Let \widehat{M} be a compact C^1 manifold without boundary. Let Ω be an open subset such that near every boundary point, Ω is locally bi-Lipschitz equivalent to a half space. Let K be a compact subset of $M = \overline{\Omega}$. Let U be a (relatively) open set in M such that $K \subset U$. Our aim is to find a compact set K' , such that $K \subset K' \subset U$ so that solutions to the appropriate mixed boundary problem for $L - \mu$ in connected components of $M \setminus K'$ are C^α up to the boundary. (This would replace Lemma 5.2 and Propositions 2.3 and 5.1 in the construction of K_ℓ in (4.1).)

To find K' given K , cover M with a finite number of coordinate charts that are each the bi-Lipschitz image of a closed cube, some of them interior to Ω and others with a boundary face on $\partial\Omega$. Fix $\varepsilon > 0$, and subdivide each closed cube of the covering dyadically to get a finite covering by cubes of diameter less than ε . Note that although this is not a disjoint covering because of the overlap of the coordinate charts, it is a finite covering. Define K' as the union of cubes in the subdivision that intersect K .

If K' consists of just one cube, this cube evidently has bi-Lipschitz boundary and clean interface with $\partial\Omega$ or is disjoint from $\partial\Omega$. We can use the reflection across $\partial\Omega$ in a single coordinate chart to see that C^α regularity reduces to an issue of C^α regularity for operators L or $L - \mu$ with bounded measurable coefficients with Dirichlet conditions on the complement of a bi-Lipschitz cube, which we know is valid by odd reflection. In general, K' is a finite union of such cubes. The even reflection across $\partial\Omega$ reduces us to the following conjectured local Dirichlet regularity lemma.

Lemma 6.1. *(Conjecture) Let A be an elliptic matrix with bounded measurable coefficients and let V (or equivalently $V - \mu$) be a bounded measurable function on the unit ball B in \mathbb{R}^n . Set $L = \operatorname{div}(A \operatorname{grad}) + V - \mu$. Let K be a finite union of bi-Lipschitz images of closed cubes and set $U = B \setminus K$. Fix $\alpha_0 > 0$ sufficiently small. Suppose that $u \in W^{1,2}(B)$, $v \in C^{\alpha_0}(B) \cap W^{1,2}(B)$, $u - v = 0$ almost everywhere on $K \cap B$, and $Lu = 0$ in the weak sense on U . Then $u \in C^\alpha(B_{1/2})$ for some $\alpha > 0$.*

We hope to prove this lemma shortly. We expect that it follows either directly from Theorem 8.29 of [GT] or from the method used to prove that boundary Hölder estimate. Technically, we don't really need C^α continuity, only continuity. But the discussion in Proposition 5.3 relied on a uniform modulus of continuity as $\varepsilon \rightarrow 0$ of eigenfunctions associated with coefficient matrices A^ε . So we prefer to prove Hölder continuity.

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E-mail address: arnold@umn.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, USA

E-mail address: guy.david@u-psud.fr

UNIV PARIS-SUD, LABORATOIRE DE MATHÉMATIQUES, CNRS, UMR 8658 ORSAY, F-91405

E-mail address: marcel.filoche@polytechnique.edu

PHYSIQUE DE LA MATIÈRE CONDENSÉE, ECOLE POLYTECHNIQUE, CNRS, PALAISEAU, FRANCE

E-mail address: jerison@math.mit.edu

MATHEMATICS DEPARTMENT, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA, USA

E-mail address: svitlana@math.umn.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, USA