New first-order formulation for the Einstein equations

Alexander M. Alekseenko*
School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

Douglas N. Arnold†
Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, Minnesota 55455, USA

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We derive a new first-order formulation for Einstein's equations which involves fewer unknowns than other first-order formulations that have been proposed. The new formulation is based on the $3+1$ decomposition with arbitrary lapse and shift. In the reduction to first-order form only eight particular combinations of the 18 first derivatives of the spatial metric are introduced. In the case of linearization about Minkowski space, the new formulation consists of a symmetric hyperbolic system in 14 unknowns, namely, the components of the extrinsic curvature perturbation and the eight new variables, from whose solution the metric perturbation can be computed by integration.

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I. INTRODUCTION

Many ways have been proposed to formulate Einstein's equations of general relativity in a manner suitable for numeric computation. In this paper we introduce a new first-order formulation for Einstein's equations. This system involves fewer unknowns than other first-order formulations that have been proposed and does not require any arbitrary parameters. In the simplest case of linearization around Minkowski space with constant lapse and vanishing shift, the system has the simple form

$$\frac{1}{\sqrt{2}}\partial_t\kappa_{ij}=\partial^l\lambda_{l(ij)},\quad \frac{1}{\sqrt{2}}\partial_t\lambda_{lji}=\partial_{[l}\kappa_{j]i}. \tag{1}$$

Here κ_{ij} is the extrinsic curvature perturbation, a symmetric tensor, and λ_{ijk} is a third-order tensor which is antisymmetric with respect to the first two indices and satisfies a cyclic identity, with the result that the system above is symmetric hyperbolic in 14 unknowns.

Our approach applies as well to the full nonlinear Arnowitt-Deser-Misner (ADM) system with arbitrary lapse and shift. We work with the actual lapse, rather than a densitized version. In the nonlinear case the system involves 20 unknowns and can be written

$$\partial_0 h_{ij}=-2ak_{ij}+2h_{s(i}\partial_{j)}b^s,$$

$$\frac{1}{\sqrt{2}}\partial_0 k_{ij}=ah_{mi}h_{nj}\partial_l f^{l(mn)}+\cdots,$$

$$\frac{1}{\sqrt{2}}\partial_0 f_{lmn}=\partial_{[l}(ak)_{m]n}+\cdots.$$

Here $\partial_0:=\partial_t-b^l\partial_l$ is the convective derivative, indices are raised and lowered using the spatial metric components h_{ij}, and the omitted terms are algebraic expressions in the h_{ij}, their spatial derivatives $\partial_l h_{ij}$, and the extrinsic curvature components k_{ij}, and also involve the lapse a and shift components b^i. The f_{lmn}, which depend on the first spatial derivatives of the spatial metric, satisfy the same symmetries as in the linear case, and so represent eight unknowns.

As is common, our derivation will start from the Arnowitt-Deser-Misner $3+1$ decomposition [1]; see also [2]. The ADM approach introduces a system of coordinates $t=x_0,x_1,x_2,x_3$, with t a timelike variable and the x_i spacelike for $i=1,2,3$, and encodes the four-metric of spacetime as a time-varying three-metric on a three-dimensional domain together with the lapse and shift, which are scalar-valued and three-vector-valued functions of time and space, respectively. Specifically, the coordinates of the four-metric are given by

$$g_{00}=-a^2+b_ib_jh^{ij},\quad g_{0i}=b_i,\quad g_{ij}=h_{ij}.$$

Here a denotes the lapse, the b_i are the components of the shift vector b, and the h_{ij} are the components of the spatial metric h. As usual, roman indices run from 1 to 3 and (h^{ij}) denotes the matrix inverse to (h_{ij}). Let D_i denote the covariant derivative operator associated with the spatial metric and set

$$k_{ij}=-\frac{1}{2a}\partial_t h_{ij}+\frac{1}{a}D_{(i}b_{j)},$$

the extrinsic curvature. Then the ADM equations for a vacuum spacetime are

$$\partial_t h_{ij}=-2ak_{ij}+2D_{(i}b_{j)}, \tag{2}$$

$$\partial_t k_{ij}=a[R_{ij}+(k_l^l)k_{ij}-2k_{il}k_j^l]+b^lD_lk_{ij}+k_{il}D_jb^l$$
$$+k_{lj}D_ib^l-D_iD_ja, \tag{3}$$

$$R_i^i+(k_i^i)^2-k_{ij}k^{ij}=0, \tag{4}$$

*Electronic address: alekseen@math.umn.edu
†Electronic address: arnold@ima.umn.edu

$$D^j k_{ij} - D_i k_j^j = 0. \tag{5}$$

Here R denotes the spatial Ricci tensor, whose components are given by second-order spatial partial differential operators applied to the spatial metric components. Also indices are raised and traces taken with respect to the spatial metric. We have used the notation of indices in parentheses to denote the symmetric part of a tensor: $D_{(i}b_{j)} := (D_i b_j + D_j b_i)/2$. Later we will similarly use bracketed indices to denote the antisymmetric part and sometimes bars to separate indices, so, as an example, $u_{[i|j|k]l} := (u_{ijkl} - u_{kjil})/2$.

Equations (4) and (5), called the Hamiltonian and momentum constraints, do not involve time differentiation. The first two equations are the evolution equations. A typical approach is to determine the lapse and shift in some way, find relevant initial data for h and k satisfying the constraint equations, and then solve the evolution equations to determine the metric and extrinsic curvature for future times. The constraint equations may or may not be explicitly imposed during the evolution. For exact solutions of the evolution equation with initial data exactly satisfying the constraints, the constraints are automatically satisfied for future times.

The system of evolution equations for h and k is first order in time and second order in space. They are not hyperbolic in any usual sense, and their direct discretization seems difficult. Therefore, many authors [3–6] have considered reformulations into more standard first-order hyperbolic systems. Typically these approaches involve introducing all the first spatial derivatives of the spatial metric components, or quantities closely related to them, as 18 additional unknowns. The resulting systems involve many variables, sometimes 30 or more. In the formulation proposed here, we introduce only eight particular combinations of the first derivatives of the metric components.

In the next section of the paper we present our approach as applied to a linearization of the ADM system. This allows us to demonstrate the basic ideas with a minimal of technical complications and to rigorously establish the relationship between the new formulation and the ADM system. In the third and final section of the paper we carry out the derivation in the case of the full nonlinear ADM system.

II. SYMMETRIC FORMULATION FOR THE LINEARIZED SYSTEM

We linearize the ADM equations about the trivial solution obtained by representing Minkowski spacetime in Cartesian coordinates: $h_{ij} = \delta_{ij}$, $k_{ij} = 0$, $a = 1$, $b_i = 0$. Consider a perturbation given by $h_{ij} = \delta_{ij} + \gamma_{ij}$, $k_{ij} = \kappa_{ij}$, $a = 1 + \alpha$, $b_i = \beta_i$, with the γ_{ij}, κ_{ij}, α, and β_i supposed to be small. Substituting these expressions into the ADM system and ignoring terms which are at least quadratic with respect to the γ_{ij}, κ_{ij}, α, and β_i and their derivatives, we obtain the linear system we shall study:

$$\partial_t \gamma_{ij} = -2\kappa_{ij} + 2\partial_{(i}\beta_{j)}, \tag{6}$$

$$\partial_t \kappa_{ij} = (P\gamma)_{ij} - \partial_i \partial_j \alpha, \tag{7}$$

$$(P\gamma)_i^i = 0, \tag{8}$$

$$\partial^j \kappa_{ij} - \partial_i \kappa_j^j = 0. \tag{9}$$

Here $P\gamma$ is the linearized Ricci tensor, with components given by

$$(P\gamma)_{ij} = \frac{1}{2}\partial_i \partial^l \gamma_{lj} + \frac{1}{2}\partial_j \partial^l \gamma_{li} - \frac{1}{2}\partial^l \partial_l \gamma_{ij} - \frac{1}{2}\partial_i \partial_j \gamma_l^l. \tag{10}$$

In these expressions, and in general when we deal with the linearized formulation, indices are raised and lowered with respect to the Euclidean metric in $\mathbb{R}^3$, so, for example, ∂^i and ∂_i are identical.

In order to reduce the linearized ADM system to first-order symmetric hyperbolic form, we first develop an identity for $P\gamma$ valid for any symmetric matrix field γ. It is useful to introduce the notations $\mathbb{S}$ for the six-dimensional space of symmetric matrices and $\mathbb{T}$ for the eight-dimensional space of triply indexed arrays (w_{ijk}) which are skew symmetric in the first two indices and satisfy the cyclic property $w_{ijk} + w_{jki} + w_{kij} = 0$.

We define the operator M: $C^\infty(\mathbb{R}^3,\mathbb{S}) \to C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ by $(Mu)_i = \partial^l u_{il} - \partial_i u_l^l$. [Note that the linearized momentum constraint (9) is simply $M\kappa = 0$.] From the definition (10), we have

$$(P\gamma)_{ij} = -\partial^l \partial_{[l}\gamma_{j]i} + \frac{1}{2}\partial_i (M\gamma)_j. \tag{11}$$

Now, for any vector-valued function v, we have

$$\partial_i v_j = \partial^l v_j \delta_{li} = 2\partial^l v_{[j}\delta_{l]i} + \partial^l v_l \delta_{ij}.$$

Applying this identity to Eq. (11) with $v = M\gamma$ we get

$$(P\gamma)_{ij} = -\partial^l [\partial_{[l}\gamma_{j]i} + (M\gamma)_{[l}\delta_{j]i}] + \frac{1}{2}\partial^l (M\gamma)_l \delta_{ij}. \tag{12}$$

Define the operators L: $C^\infty(\mathbb{R}^3,\mathbb{S}) \to C^\infty(\mathbb{R}^3,\mathbb{T})$ and L^*: $C^\infty(\mathbb{R}^3,\mathbb{T}) \to C^\infty(\mathbb{R}^3,\mathbb{S})$ by

$$(Lu)_{lji} = \partial_{[l}u_{j]i}, \quad (L^* w)_{ij} = -\partial^l w_{l(ij)}.$$

One easily verifies that operators L and L^* are formal adjoints to each other with respect to the scalar products $\langle u,v\rangle = \int u_{pq}v^{pq}dx$ and $\langle z,w\rangle = \int z_{pqr}w^{pqr}dx$ in the spaces $C^\infty(\mathbb{R}^3,\mathbb{S})$ and $C^\infty(\mathbb{R}^3,\mathbb{T})$ respectively. Introducing

$$\lambda_{lji} = -\frac{1}{\sqrt{2}}[(L\gamma)_{lji} + (M\gamma)_{[l}\delta_{j]i}], \tag{13}$$

we can then restate Eq. (12) as

$$(P\gamma)_{ij} = \sqrt{2}\partial^l \lambda_{lji} + \frac{1}{2}\partial^l (M\gamma)_l \delta_{ij}.$$

(The reason for the factor of $\sqrt{2}$ will become apparent shortly.) Taking symmetric parts, the last equation becomes

$$(P\gamma)_{ij} = -\sqrt{2}(L^*\lambda)_{ij} + \frac{1}{2}\partial^l(M\gamma)_l\delta_{ij}. \tag{14}$$

We are now ready to introduce our first-order symmetric hyperbolic formulation. The unknowns will be $\kappa \in C^\infty(\mathbb{R}^3,\mathbb{S})$ and $\lambda \in C^\infty(\mathbb{R}^3,\mathbb{T})$, so the system has 14 independent variables in all. Substituting Eq. (14) in Eq. (7) and noting that $\partial^l(M\gamma)_l = (P\gamma)^l_l = 0$ by the linearized Hamiltonian constraint (8), we obtain an evolution equation for κ:

$$\partial_t\kappa_{ij} = -\sqrt{2}(L^*\lambda)_{ij} - \partial_i\partial_j\alpha. \tag{15}$$

To obtain an evolution equation for λ, we differentiate Eq. (13) with respect to time and substitute Eq. (6) to eliminate γ. Simplifying and using the linearized momentum constraint (9) we obtain

$$\partial_t\lambda_{lji} = \sqrt{2}(L\kappa)_{lji} - \tau_{lji}, \tag{16}$$

where τ can be determined from $(\epsilon\beta)_{ij} := \partial_{(i}\beta_{j)}$ by

$$\tau_{lji} = \sqrt{2}[(L\epsilon\beta)_{lji} + (M\epsilon\beta)_{[l}\delta_{j]i}]$$
$$= \frac{1}{\sqrt{2}}(\partial_i\partial_{[l}\beta_{j]} + \partial^m\partial_{[m}\beta_{l]}\delta_{ij} - \partial^m\partial_{[m}\beta_{j]}\delta_{il}).$$

Equations (15) and (16) constitute a first-order symmetric hyperbolic system (this is clear, since L and L^* are formal adjoints). It follows that (see, e.g., [7], Sec. 7.3.2), if the lapse and the shift are sufficiently smooth, then for arbitrary initial values $\kappa(0)$ and $\lambda(0)$ belonging to $H^1(\mathbb{R}^3)$, there exists a unique solution to the system (15) and (16) with components in $H^1((0,T)\times\mathbb{R}^3)$.

The Cauchy problem for the original linearized ADM system consists of Eqs. (6)–(9) together with specific initial values $\gamma(0)$ and $\kappa(0)$. The foregoing derivation shows that, if γ and κ satisfy the ADM system and λ is defined by Eq. (13), then κ and λ satisfy the symmetric hyperbolic system (15), (16). Conversely, to recover the solution to the ADM system from (15), (16), the same initial condition should be imposed on κ and λ should be taken initially to be

$$\lambda_{lji}(0) = -\frac{1}{\sqrt{2}}[(L\gamma(0))_{lji} + (M\gamma(0))_{[l}\delta_{j]i}]. \tag{17}$$

Once κ and λ are determined, the metric perturbation γ is given by

$$\gamma_{ij} = \gamma_{ij}(0) - 2\int_0^t(\kappa_{ij} - \partial_{(i}\beta_{j)}), \tag{18}$$

as follows from Eq. (6).

Theorem 1. Let the lapse perturbation α and shift perturbation β be given. Suppose that initial data $\gamma(0)$ and $\kappa(0)$ are specified satisfying the constraint equations (8),(9) at time $t = 0$. Define $\lambda(0)$ by Eq. (17), and determine κ and λ from the first-order symmetric hyperbolic system (15),(16). Finally, define γ by Eq. (18). Then the ADM system (6)–(9) is satisfied.

Proof. Equation (6) follows from Eq. (18) by differentiation.

Next we verify the momentum constraint (9). To do so we will show that $\mu := M\kappa$ satisfies a second-order wave equation, and that $\mu(0) = \partial_t\mu(0) = 0$. Indeed, $\mu(0) = 0$ by assumptions. To see that $\partial_t\mu(0) = 0$, we apply the operator M to Eq. (15) and use the fact that M annihilates the Hessian $\partial_i\partial_j\alpha$ for any function α. Therefore $\partial_t\mu = -\sqrt{2}ML^*\lambda$. Using Eq. (14) and the assumption that γ satisfies the Hamiltonian constraint at the initial time, we find that

$$\partial_t\mu_i(0) = (MP\gamma(0))_i = -\frac{1}{2}\partial_i\partial^l(M\gamma(0))_l = 0.$$

To obtain a second-order equation for μ, first we differentiate Eq. (15) in time and substitute Eq. (16) to get a second-order equation for κ:

$$\partial_t^2\kappa_{ij} = -2(L^*L\kappa_{ij}) - \partial_i\partial_j\alpha.$$

Here we have used the fact that $L^*\tau \equiv 0$. Apply M to the last equation. Using the identity $(ML^*L\kappa) = -\partial^l\partial_{(i}(M\kappa)_{l)}/2$ and the fact that M annihilates Hessians, we find that $\mu := M\kappa$ satisfies the second-order hyperbolic equation

$$\partial_t^2\mu_i = \partial^l\partial_{(i}\mu_{l)}.$$

This is simply an elastic wave equation. Since $\mu(0) = \partial_t\mu(0) = 0$, μ vanishes for all time, i.e., the momentum constraint is satisfied.

Now $(P\kappa)^i_i = \partial^i(M\kappa)_i = 0$. Moreover, P applied to $\epsilon\beta$ is identically zero. Therefore if we apply P to Eq. (18) and take the trace, we find that $(P\gamma)^i_i = (P\gamma(0))^i_i$, which vanishes by assumption. This verifies the Hamiltonian constraint (8).

It remains to verify Eq. (7) which, in view of Eq. (15), comes down to showing that $\sqrt{2}L^*\lambda = -P\gamma$. Since we have verified the Hamiltonian constraint, this will follow if we can establish Eq. (14), which is itself a consequence of Eq. (13). We used Eq. (13) at time $t = 0$ to initialize λ, so it is sufficient to show that

$$\partial_t\lambda_{lji} = -\frac{1}{\sqrt{2}}\partial_t[(L\gamma)_{lji} + (M\gamma)_{[l}\delta_{j]i}].$$

This follows directly from Eqs. (16) and (6). ■

We conclude this section by computing the plane wave solutions to the hyperbolic system (1). That is, we seek solutions of the form

$$\kappa_{ij} = \tilde{\kappa}_{ij}f(st - n^a x_a), \quad \lambda_{lji} = \tilde{\lambda}_{lji}f(st - n^a x_a),$$

where the real number s gives the wave speed and the unit vector n_i the wave direction, the polarizations $\tilde{\kappa}_{ij}$ and $\tilde{\lambda}_{lji}$ are constant, and the profile $f(t)$ is an arbitrary differentiable function.

Substituting these expressions into Eq. (1), we get

$$s\tilde{\kappa}_{ij}f'(st-n^a x_a) = -\sqrt{2}n^l\tilde{\lambda}_{l(ij)}f'(st-n^a x_a),$$

$$s\tilde{\lambda}_{lji}f'(st-n^a x_a) = -\sqrt{2}n_{[l}\tilde{\kappa}_{j]i}f'(st-n^a x_a),$$

and so the system reduces to the following linear eigenvalue problem:

$$s\tilde{\kappa}_{ij} = -\sqrt{2}n^l\tilde{\lambda}_{l(ij)}, \tag{19}$$

$$s\tilde{\lambda}_{lji} = -\sqrt{2}n_{[l}\tilde{\kappa}_{j]i}, \tag{20}$$

with the wave speed s as eigenvalues and the pairs $(\tilde{\kappa}_{ij}, \tilde{\lambda}_{lji})$ as eigenvectors. The eigenvalues of this system are 0 (multiplicity 4), ± 1 (each multiplicity 3), and $\pm 1/\sqrt{2}$ (each multiplicity 2). To verify this and describe the eigenvectors we introduce a unit vector m_i perpendicular to n_i, and set $l_i = \varepsilon_i^{ab}n_a m_b$ to complete an orthonormal frame. Then the following solution to the eigenvalue problem can be checked by direct substitution into Eqs. (19),(20):

$$s=0: \quad (0, m_{[l}l_{j]}m_i), \quad (0, m_{[l}l_{j]}l_i), \quad (n_i n_j, 0),$$

$$(0, 2m_{[l}l_{j]}n_i + n_{[l}l_{j]}m_i - n_{[l}m_{j]}l_i);$$

$$s=\pm 1: \quad (l_i l_j - m_i m_j, \mp\sqrt{2}(n_{[l}l_{j]}l_i - n_{[l}m_{j]}m_i)),$$

$$(l_i m_j + m_i l_j, \mp\sqrt{2}(n_{[l}l_{j]}m_i + n_{[l}m_{j]}l_i)),$$

$$(l_i l_j + m_i m_j, \mp\sqrt{2}(n_{[l}l_{j]}l_i + n_{[l}m_{j]}m_i));$$

$$s=\pm 1/\sqrt{2}: \quad (n_{(i}l_{j)}, \mp n_{[l}l_{j]}n_i), \quad (n_{(i}m_{j)}, \mp n_{[l}m_{j]}n_i).$$

III. DECOMPOSITION OF THE FULL ADM SYSTEM

In this section we develop a first-order formulation of the full nonlinear ADM system analogous to that developed for the linearized system in Sec. II. We continue to assume that the underlying manifold is topologically $\mathbb{R}^3$ and view the ADM system as equations for the evolution of a Riemannian three-metric h on $\mathbb{R}^3$. Thus h_{ij}, $1 \leqslant i,j \leqslant 3$ are the components of the spatial metric. They form a positive-definite symmetric matrix defined at each point of $\mathbb{R}^3$ and varying in time. Indices on other fields are lowered and raised using h_{ij} and the inverse matrix field h^{ij}.

For the components of the Riemann tensor we have

$$R_{ijkl} = \partial_j\Gamma_{ikl} - \partial_i\Gamma_{jkl} + h^{mn}(\Gamma_{jkm}\Gamma_{iln} - \Gamma_{ikm}\Gamma_{jln}),$$

where the Cristoffel symbols are defined by $\Gamma_{ijk} = (\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij})/2$. The components of the Ricci tensor are given by $R_{ij} = h^{pq}R_{piqj}$, which yields

$$R_{ij} = \frac{1}{2}h^{pq}(\partial_p\partial_j h_{iq} + \partial_i\partial_p h_{qj} - \partial_p\partial_q h_{ij} - \partial_i\partial_j h_{pq})$$
$$+ h^{pq}h^{rs}(\Gamma_{ips}\Gamma_{qjr} - \Gamma_{pqs}\Gamma_{ijr}).$$

We define a second-order linear partial differential operator P: $C^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3}) \to C^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ by

$$(Pu)_{ij} = \frac{1}{2}[\partial_p(h^{pq}\partial_j u_{qi}) + \partial_i(h^{pq}\partial_p u_{jq}) - \partial_p(h^{pq}\partial_q u_{ji})$$
$$- \partial_i(h^{pq}\partial_j u_{pq})].$$

Note that, in the case $h_{ij} = \delta_{ij}$, P coincides with the linearized Ricci operator introduced in Eq. (10). In the nonlinear case, P is related to the Ricci tensor by the equation

$$R_{ij} = (Ph)_{ij} + c_{ij}^{1)}, \tag{21}$$

where

$$c_{ij}^{1)} := -\frac{1}{2}[(\partial_p h^{pq})(\partial_j h_{iq}) + (\partial_i h^{pq})(\partial_p h_{qj}) - (\partial_p h^{pq})(\partial_q h_{ij})$$
$$- (\partial_i h^{pq})(\partial_j h_{pq})] + h^{pq}h^{rs}(\Gamma_{ips}\Gamma_{qjr} - \Gamma_{pqs}\Gamma_{ijr})$$

is a first-order differential operator in h_{ij}.

Define an operator M: $C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by

$$(Mu)_i = 2h^{pq}\partial_{[p}u_{i]q}.$$

This formula extends the definition of the linearized momentum constraint operator introduced in the previous section. Up to lower order terms the momentum constraint (5) is given by the vanishing of Mk:

$$D^p k_{ip} - D_i k_p^p = 2h^{pq}D_{[p}k_{i]q} = (Mk)_i - 2h^{pq}\Gamma^r_{q[p}k_{i]r} \tag{22}$$

(where $\Gamma^r_{qp} := h^{rs}\Gamma_{qps}$).

Finally, we introduce operators L: $C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{T})$ and L^*: $C^\infty(\mathbb{R}^3, \mathbb{T}) \to C^\infty(\mathbb{R}^3, \mathbb{S})$ by

$$(Lu)_{ijl} = \partial_{[i}u_{j]l}, \quad (L^*v)_{ij} = -h_{qi}h_{rj}\partial_p v^{p(qr)}.$$

As in the linear case, the operators L and L^* are formal adjoints to each other with respect to the scalar products $\langle u, w\rangle = \int u_{ij}w^{ij}\mathrm{d}x$ and $\langle v, z\rangle = \int v_{ijl}z^{ijl}\mathrm{d}x$ on the spaces $C^\infty(\mathbb{R}^3, \mathbb{S})$ and $C^\infty(\mathbb{R}^3, \mathbb{T})$, respectively.

Finally, we introduce the new variables

$$f_{lmn} = -\frac{1}{\sqrt{2}}[(Lh)_{lmn} + (Mh)_{[l}h_{m]n}], \tag{23}$$

and develop the analogue of the identity (14).

Lemma 1. The following identity is valid for the Ricci tensor:

$$R_{ij} = -\sqrt{2}(L^*f)_{ij} + \frac{1}{2}\partial_p[h^{pq}(Mh)_q h_{ij}] + c_{ij}^{2)}, \tag{24}$$

where

$$c_{ij}^{2)} = c_{(ij)}^{1)} + h_{qi}h_{rj}[\partial_p(h^{qm}h^{rn})]h^{pl}\Bigg[(Lh)_{l(mn)}$$
$$+ \frac{1}{2}(Mh)_{[l}h_{m]n} + \frac{1}{2}(Mh)_{[l}h_{n]m}\Bigg]$$

is first order in h_{ij}.

Proof. The formula (24) is a consequence of the identity

$$-\sqrt{2}(L^*f)_{ij}=(Ph)_{(ij)}-\frac{1}{2}\partial_p[h^{pq}(Mh)_q h_{ij}]+c^{3)}_{ij}, \tag{25}$$

where

$$c^{3)}_{ij}=-h_{qi}h_{rj}[\partial_p(h^{qm}h^{rn})]h^{pl}\left[(Lh)_{l(mn)}+\frac{1}{2}(Mh)_{[l}h_{m]n}+\frac{1}{2}(Mh)_{[l}h_{n]m}\right].$$

To prove the identity (25) we note, first, that the operator P can be rewritten in terms of operators L and M as

$$(Pu)_{ij}=-\partial_p[h^{pq}(Lu)_{qji}]+\frac{1}{2}\partial_i(Mu)_j,$$

which yields

$$(Pu)_{(ij)}=-\partial_p[h^{pq}(Lu)_{q(ij)}]+\frac{1}{2}\partial_{(i}(Mu)_{j)},$$

and, second, that according to the definition of L^*,

$$\begin{aligned}(L^*v)_{ij}&=-h_{qi}h_{rj}\partial_p(h^{pl}h^{qm}h^{rn}v_{l(mn)})\\&=-\partial_p(h^{pq}v_{q(ij)})-h_{qi}h_{rj}[\partial_p(h^{qm}h^{rn})]h^{pl}v_{l(mn)}.\end{aligned} \tag{26}$$

To derive the identity (25) we substitute $-\sqrt{2}f_{qij}$ for v_{qij} in Eq. (26). The first term on the right-hand side of Eq. (26) then becomes

$$\begin{aligned}&-\partial_p\left(h^{pq}\left[(Lh)_{q(ij)}+\frac{1}{2}(Mh)_q h_{ij}-\frac{1}{2}(Mh)_{(i}h_{j)q}\right]\right)\\&=-\partial_p[h^{pq}(Lh)_{q(ij)}]-\frac{1}{2}\partial_p[h^{pq}(Mh)_q h_{ij}]\\&\quad+\frac{1}{2}\partial_p[h^{pq}(Mh)_{(i}h_{j)q}]\\&=-\partial_p[h^{pq}(Lh)_{q(ij)}]-\frac{1}{2}\partial_p[h^{pq}(Mh)_q h_{ij}]+\frac{1}{2}\partial_{(i}(Mh)_{j)}\\&=(Ph)_{(ij)}-\frac{1}{2}\partial_p[h^{pq}(Mh)_q h_{ij}].\end{aligned}$$

The substitution of $-\sqrt{2}f_{lmn}$ for v_{lmn} into the second term of Eq. (26) gives the term $c^{3)}_{ij}$ precisely. The rest of the proof follows from the identity (21). ∎

We now proceed to the derivation of the new formulation of the ADM system. In Eq. (3) we substitute $\partial_0+b^l\partial_l$ for ∂_t and replace R_{ij} with the right side of Eq. (24) to get

$$\partial_0 k_{ij}=-\sqrt{2}a(L^*f)_{ij}+\frac{1}{2}a\partial_p[h^{pq}(Mh)_q h_{ij}]+c^{4)}_{ij}, \tag{27}$$

where

$$c^{4)}_{ij}=a[c^{2)}_{ij}+(k^l_l)k_{ij}-2k_{il}k^l_j]+k_{il}\partial_j b^l+k_{lj}\partial_i b^l-D_iD_j a.$$

Here we used the fact that the Lie derivative $b^lD_lk_{ij}+k_{il}D_jb^l+k_{lj}D_ib^l=b^l\partial_lk_{ij}+k_{il}\partial_jb^l+k_{lj}\partial_ib^l$. We treat the second term on the right-hand side of Eq. (24) using the Hamiltonian constraint (4). Now,

$$\begin{aligned}R^i_i&=\frac{1}{2}h^{ij}h^{pq}(\partial_p\partial_jh_{iq}+\partial_i\partial_ph_{qj}-\partial_p\partial_qh_{ij}-\partial_i\partial_jh_{pq})\\&\quad+h^{ij}h^{pq}h^{rs}(\Gamma_{ips}\Gamma_{qjr}-\Gamma_{pqs}\Gamma_{ijr})\\&=h^{ij}h^{pq}(\partial_i\partial_ph_{qj}-\partial_i\partial_jh_{pq})+h^{ij}h^{pq}h^{rs}\\&\quad\times(\Gamma_{ips}\Gamma_{qjr}-\Gamma_{pqs}\Gamma_{ijr})\\&=\partial_i[h^{ij}h^{pq}(\partial_ph_{qj}-\partial_jh_{pq})]-[\partial_i(h^{ij}h^{pq})](\partial_ph_{qj}-\partial_jh_{pq})\\&\quad+h^{ij}h^{pq}h^{rs}(\Gamma_{ips}\Gamma_{qjr}-\Gamma_{pqs}\Gamma_{ijr})\\&=\partial_i[h^{ij}(Mh)_j]+c^{5)},\end{aligned}$$

where

$$\begin{aligned}c^{5)}=&-[\partial_i(h^{ij}h^{pq})](\partial_ph_{qj}-\partial_jh_{pq})+h^{ij}h^{pq}h^{rs}\\&\times(\Gamma_{ips}\Gamma_{qjr}-\Gamma_{pqs}\Gamma_{ijr}).\end{aligned}$$

Hence, due to the Hamiltonian constraint,

$$\begin{aligned}\partial_p[h^{pq}(Mh)_qh_{ij}]&=\{\partial_p[h^{pq}(Mh)_q]\}h_{ij}+h^{pq}(Mh)_q(\partial_ph_{ij})\\&=[k_{pq}k^{pq}-(k^p_p)^2-c^{5)}]h_{ij}\\&\quad+h^{pq}(Mh)_q(\partial_ph_{ij}).\end{aligned}$$

Combining all lower order terms into an expression B_{ij}, first order in h, we reduce Eq. (27) to

$$\partial_0k_{ij}=-\sqrt{2}a(L^*f)_{ij}+B_{ij}. \tag{28}$$

This is the first evolution equation of our system.

The second evolution equation will be obtained by applying ∂_0 to the definition of f (23):

$$\partial_0f_{lmn}=-\frac{1}{\sqrt{2}}\{\partial_0(Lh)_{lmn}+\partial_0[(Mh)_{[l}h_{m]n}]\}.$$

First, we note that

$$\partial_0(Lh)_{lmn}=(L\partial_0h)_{lmn}+\frac{1}{2}[(\partial_lb^s)(\partial_sh_{mn})-(\partial_mb^s)(\partial_sh_{ln})].$$

Using the fact that $(Mu)_l=2h^{pq}(Lu)_{plq}$, we then get

$$\partial_0(Mh)_l=(M\partial_0 h)_l+2(\partial_0 h^{pq})(Lh)_{plq}+h^{pq}[(\partial_p b^s)(\partial_s h_{lq}) -(\partial_l b^s)(\partial_s h_{pq})].$$

If we use this formula to compute $\partial_0[(Mh)_l h_{mn}]$ and then antisymmetrize in l and m, we obtain

$$\partial_0 f_{lmn}=-\frac{1}{\sqrt{2}}[(L\partial_0 h)_{lmn}+(M\partial_0 h)_{[l}h_{m]n}]-\sqrt{2}(\partial_0 h^{pq})\times(Lh)_{p[l|q|}h_{m]n}-\frac{1}{\sqrt{2}}(Mh)_{[l}\partial_0 h_{m]n}+c^{6)}_{lmn}, \quad (29)$$

where

$$c^{6)}_{lmn}=-\frac{1}{2\sqrt{2}}[(\partial_l b^s)(\partial_s h_{mn})-(\partial_m b^s)(\partial_s h_{ln})]-\frac{1}{\sqrt{2}}h^{pq}[(\partial_p b^s)(\partial_s h_{q[l})h_{m]n}-(\partial_s h_{pq})(\partial_{[l} b^s)h_{m]n}].$$

Next we use Eq. (2) to relate the terms in Eq. (29) involving $\partial_0 h$ to the extrinsic curvature k. For the Lie derivative of the metric, we have

$$2D_{(i}b_{j)}=b^s\partial_s h_{ij}+2h_{s(i}\partial_{j)}b^s.$$

Using this, Eq. (2) becomes

$$\partial_0 h_{ij}=-2ak_{ij}+2w_{ij}, \quad (30)$$

where $w_{ij}:=h_{s(i}\partial_{j)}b^s$. Using the Leibniz rule we can then verify that

$$\partial_0 h^{ij}=2ak^{ij}-2w^{ij}.$$

Substituting these expressions in Eq. (29) we obtain

$$\partial_0 f_{lmn}=\sqrt{2}[L(ak)]_{lmn}+\sqrt{2}[M(ak)]_{[l}h_{m]n}+c^{7)}_{lmn}, \quad (31)$$

where

$$c^{7)}_{lmn}=c^{6)}_{lmn}-\sqrt{2}[(Lw)_{lmn}+(Mw)_{[l}h_{m]n}]-2\sqrt{2}[ak^{pq}-w^{pq}]\times(Lh)_{p[l|q|}h_{m]n}+\sqrt{2}(Mh)_{[l}ak_{m]n}-\sqrt{2}(Mh)_{[l}w_{m]n}.$$

The final step is to invoke the momentum constraint to simplify the second term on the right-hand side of Eq. (31). Indeed, since the right-hand side of Eq. (22) vanishes,

$$[M(ak)]_l=a(Mk)_l+2h^{pq}(\partial_{[p}a)k_{l]q}=2ah^{pq}\Gamma^s_{q[p}k_{l]s}+2h^{pq}(\partial_{[p}a)k_{l]q}.$$

Substituting this in Eq. (31), we obtain the desired second evolution equation:

$$\partial_0 f_{lmn}=\sqrt{2}[L(ak)]_{lmn}+C_{lmn}, \quad (32)$$

where

$$C_{lmn}=c^{7)}_{lmn}+\sqrt{2}[ah^{pq}\Gamma^s_{q[p}k_{l]s}h_{mn}-ah^{pq}\Gamma^s_{q[p}k_{m]s}h_{ln}+h^{pq}(\partial_{[p}a)k_{l]q}h_{mn}-h^{pq}(\partial_{[p}a)k_{m]q}h_{ln}].$$

The two equations (28) and (32) constitute a first-order system for the unknowns k_{ij} and f_{lmn}. This system is coupled to the ordinary differential equation (30) through the terms B_{ij} and C_{lmn} which are algebraic combinations of h_{ij}, $\partial_l h_{ij}, k_{ij}$, the lapse a and the shift b, and their spatial derivatives. The foregoing derivation shows that if h and k satisfy the ADM system (2)–(5), then h, k, and f satisfy the system (30),(28),(32).

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