Spaces of Finite Element Differential Forms

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Abstract We discuss the construction of finite element spaces of differential forms which satisfy the crucial assumptions of the finite element exterior calculus, namely that they can be assembled into subcomplexes of the de Rham complex which admit commuting projections. We present two families of spaces in the case of simplicial meshes, and two other families in the case of cubical meshes. We make use of the exterior calculus and the Koszul complex to define and understand the spaces. These tools allow us to treat a wide variety of situations, which are often treated separately, in a unified fashion.

1 Introduction

The gradient, curl, and divergence are the most fundamental operators of vector calculus, appearing throughout the differential equations of mathematical physics and other applications. The finite element solution of such equations requires finite element subspaces of the natural Hilbert space domains of these operators, namely H^1, $H(\text{curl})$, and $H(\text{div})$. The construction of subspaces with desirable properties has been an active research topic for half a century. Exterior calculus provides a framework in which these fundamental operators and spaces are unified and generalized, and their properties and inter-relations clarified. Each of the operators is viewed as a particular case of the exterior derivative operator $d = d^k$ taking differential k-forms on some domain $\Omega \subset \mathbb{R}^n$ to differential $(k+1)$-forms. We regard d^k as an unbounded operator between the Hilbert spaces $L^2\Lambda^k$ and $L^2\Lambda^{k+1}$ consisting of differential forms with L^2 coefficients. The domain of d^k is the Hilbert space

$$H\Lambda^k = \left\{ u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \right\}, \tag{1.1}$$

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and all the d^k and their domains combine to form the L^2 de Rham complex

$$0 \to H\Lambda^0 \xrightarrow{d^0} H\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n \to 0.$$

Differential 0-forms and n-forms may be identified simply with functions on Ω and differential 1-forms and $(n-1)$-forms may be identified with vector fields. In three dimensions, we may use these proxies to write the de Rham complex as

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \to 0.$$

The finite element exterior calculus (FEEC) is a theory developed in the last decade [1, 5, 6] which enables the development and analysis of finite element spaces of differential forms. One major part of FEEC is carried out in the framework of Hilbert complexes, of which the L^2 de Rham complex is the most canonical example. One important outcome of FEEC is the realization that the finite dimensional subspaces $\Lambda_h^k \subset H\Lambda^k$ used in Galerkin discretizations of a variety of differential equations involving differential k-forms should satisfy two basic assumptions, beyond the obvious requirement that the spaces have good approximation properties. The first assumption is that the subspaces form a *subcomplex* of the de Rham complex, i.e., that $d\Lambda_h^k \subset \Lambda_h^{k+1}$. The second is that there exist projection operators π_h^k from $H\Lambda^k$ to Λ_h^k which commute with d in the sense that the following diagram commutes:

$$\begin{array}{ccccccccc}
H\Lambda^0 & \xrightarrow{d} & H\Lambda^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^{n-1} & \xrightarrow{d} & H\Lambda^n \\
\downarrow \pi_h^0 & & \downarrow \pi_h^1 & & & & \downarrow \pi_h^{n-1} & & \downarrow \pi_h^n \\
\Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda_h^{n-1} & \xrightarrow{d} & \Lambda_h^n
\end{array}$$

The second major part of FEEC, into which the present exposition falls, is concerned with the construction of specific finite element spaces Λ_h^k of differential forms. A special role is played by two families of finite element spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, defined for any dimension n, any simplicial mesh $\mathcal{T}_h$, any polynomial degree $r \ge 1$, and any form degree $0 \le k \le n$. Both these spaces are subspaces of $H\Lambda^k(\Omega)$. The $\mathcal{P}_r^-\Lambda^k$ spaces with increasing k and constant r form a subcomplex of L^2 de Rham complex which admits commuting projections. The same is true of the $\mathcal{P}_r\Lambda^k$ family, except in that case the polynomial degree r decreases as the form degree k increases.

We also discuss cubical meshes. In this case, there is a well-known family of elements, denoted by $\mathcal{Q}_r^-\Lambda^k$ in our notation, obtained by a tensor product construction. As for the $\mathcal{P}_r^-\Lambda^k$ family, the $\mathcal{Q}_r^-\Lambda^k$ spaces with constant degree r combine to form a de Rham subcomplex with commuting projections. We also discuss a recently discovered second family on cubical meshes, the $\mathcal{S}_r\Lambda^k$ family of [3]. Like the $\mathcal{P}_r\Lambda^k$ family, the de Rham subcomplexes for this family are obtained with decreasing degree. Moreover for large r, the $\dim \mathcal{S}_r\Lambda^k(\mathcal{T}_h)$ is much smaller dimension than $\dim \mathcal{Q}_r^- \Lambda^k$. The finite element subspaces of H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ from this family in three dimensions are new.

The remainder of the paper is organized as follows. In the next section we cover some preliminary material (which the more expert reader may wish to skip). We recall the construction of finite element spaces from spaces of shape functions and unisolvent degrees of freedom. To illustrate we discuss the Lagrange elements and carry out the proof of unisolvence in a manner that will guide our treatment of differential form spaces of higher degree. We also give a brief summary of those aspects of exterior calculus most relevant to us. In Sect. 3 we discuss the two primary families of finite element spaces for differential forms on simplicial meshes mentioned above. A key role is played by the Koszul complex, which is introduced in this section. Then, in Theorem 3.5, we give a proof of unisolvence for the $\mathcal{P}_r^-$ family which we believe to be simpler than has appeared heretofore (a similar proof could be given for the $\mathcal{P}_r$ family as well). In the final section we review the two families mentioned for cubical meshes, including a description, without proofs, of the recently discovered $\mathcal{S}_r$ family.

2 Preliminaries

2.1 The Assembly of Finite Element Spaces

Recalling the definition of a finite element space [11], we assume that the domain $\Omega \subset \mathbb{R}^n$ is triangulated by finite elements, i.e., its closure is the union of a finite set $\mathcal{T}_h$ of closed convex polyhedral elements with nonempty interiors such that the intersection of any two elements is either empty or is a common face of each of some dimension. We denote by $\Delta_d(T)$ the set of faces of T of dimensions d, so, for example, $\Delta_0(T)$ is the set of vertices of T, and $\Delta_n(T)$ is the singleton set whose only element is T. We also define $\Delta(T) = \bigcup_{0 \le d \le n} \Delta_d(T)$, the set of all faces of T. In this paper we consider the two cases of simplicial elements, in which each element T of the triangulation is an n-simplex, and cubical elements, in which element is an n-box (i.e., the Cartesian product of n intervals). To define a finite element space $\Lambda_h^k \subset H\Lambda^k(\Omega)$, we must supply, for each element $T \in \mathcal{T}_h$,

(1) A finite dimensional space $V(T)$, called the space of *shape functions*, consisting of differential k-forms on T with polynomial coefficients. The finite element space will consist of functions u which belong to the shape function spaces piecewise in the sense that $u|_T \in V(T)$ for all $T \in \mathcal{T}_h$ (allowing the possibility that u is multiply-valued on faces of dimension $< n$).
(2) A set of functionals $V(T) \to \mathbb{R}$, called the *degrees of freedom*, which are *unisolvent* (i.e., which form a basis for the dual space $V(T)^*$) and such that each degree of freedom is associated to a specific face of $f \in \Delta(T)$.

It is assumed that when two distinct elements T_1 and T_2 intersect in a common face f, the degrees of freedom of T_1 and T_2 which are associated to f are in a specific 1-to-1 correspondence. If u is a function which belongs to the shape function spaces piecewise, then we say that the degrees of freedom are single-valued on u if whenever two elements $T_1 \neq T_2$ meet in a common face, then the corresponding degrees of freedom associated to the face take the same value on $u|_{T_1}$ and $u|_{T_2}$, respectively. With these ingredients, the finite element space Λ_h^k associated to the choice of triangulation $\mathcal{T}_h$, the shape function spaces $V(T)$, and the degrees of freedom, is defined as the set of all k-forms on Ω which belong to the shape function spaces piecewise and for which all the degrees of freedom are single-valued.

The choice of the degrees of freedom associated to faces of dimension $d < n$ determine the interelement continuity imposed on the finite element subspace. The use of degrees of freedom to specify the continuity, rather than imposing the continuity a priori in the definition of the finite element space, is of great practical significance in that it assures that the finite element space can be implemented efficiently. The dimension of the space is known (it is just the sum over the faces of the triangulation of the number of degrees of freedom associated to the face) and it depends only on the topology of the triangulation, not on the coordinates of the element vertices. Moreover, the degrees of freedom lead to a computable basis for Λ_h^k in which each basis element is associated to one degree of freedom. Further, the basis is local, in that the basis element for a degree of freedom associated to a face f is nonzero only on the elements that contain f.

The finite element space so defined does not depend on the specific choice of degrees of freedom in $V(T)^*$, but only on the span of the degrees of freedom associated to each face f of T, and we shall generally specify only the span, rather than a specific choice of basis for it.

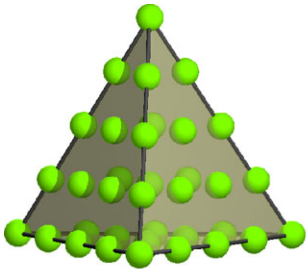
2.2 The Lagrange Finite Element Family

To illustrate these definitions and motivate the constructions for differential forms, we consider the simplest example, the Lagrange family of finite element subspaces of $H^1 = H\Lambda^0$. The Lagrange space, which we denote $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ in anticipation of its generalization below, is defined for any simplicial triangulation $\mathcal{T}_h$ in $\mathbb{R}^n$ and any polynomial degree $r \geq 1$. The shape function space is $V(T) = \mathcal{P}_r(T)$, the space of all polynomial functions on T of degree at most r. For a face f of T of dimension d, the span of the associated degrees of freedom are the functionals

$$u \in \mathcal{P}_r(T) \mapsto \int_f (\operatorname{tr}_f u)q, \quad q \in \mathcal{P}_{r-d-1}(f),\ f \in \Delta(T). \tag{2.1}$$

In interpreting this, we understand the space $\mathcal{P}_s(f)$ to be the space $\mathbb{R}$ of constants if f is 0-dimensional (a single vertex) and $s \geq 0$. Also the space $\mathcal{P}_s(f) = 0$ if $s < 0$ and f is arbitrary. The notation $\operatorname{tr}_f u$ denotes the trace of u on f, i.e., its restriction. Thus there is one degree of freedom associated to each vertex v, namely the evaluation functional $u \mapsto u(v)$. For $r \geq 2$ there are also degrees of freedom associated to

Fig. 1 Degrees of freedom for the Lagrange quartic space $\mathcal{P}_4\Lambda^0$ in 3 dimensions



the edges e of T, namely the moments of u on the edge of degree at most $r-2$:

$$u \mapsto \int_e (\operatorname{tr}_e u)q, \quad q \in \mathcal{P}_{r-2}(e).$$

For $r \geq 3$ there are degrees of freedom associated to the 2-faces, namely moments of degree at most $r-3$, etc. This is often indicated in a degree of freedom diagram, like that of Fig. 1, in which the number of symbols drawn in the interior of a face is equal to the number of degrees of freedom associated to the face.

A requirement of the definition of a finite element space is that the degrees of freedom be unisolvent. We present the proof for Lagrange elements in detail, since it will guide us when it comes to verifying unisolvence for more complicated spaces.

Theorem 2.1 (Unisolvence for the Lagrange elements) *For any $r \geq 1$ and any n-simplex T, the degrees of freedom* (2.1) *are unisolvent on $V(T) = \mathcal{P}_r(T)$.*

Proof It suffices to verify, first, that the number of degrees of freedom proposed for T does not exceed $\dim V(T)$, and, second, that if all the degrees of freedom vanish when applied to some $u \in V(T)$, then $u \equiv 0$. For the first claim, we have by (2.1) that the total number of degrees of freedom is at most

$$\sum_{d=0}^{n} \#\Delta_d(T) \dim \mathcal{P}_{r-d-1}(\mathbb{R}^d) = \sum_{d=0}^{n} \binom{n+1}{d+1}\binom{r-1}{d} = \binom{n+r}{n} = \dim \mathcal{P}_r(T),$$

where the second equality is a binomial identity which comes from expanding in the equation $(1+x)^{n+1}(1+x)^{r-1} = (1+x)^{n+r}$ and comparing the coefficients of x^n on both sides.

We prove the second claim by induction on the dimension n, the case $n=0$ being trivial. Suppose that $u \in \mathcal{P}_r(T)$ for some simplex T of dimension n and that all the degrees of freedom in (2.1) vanish. We wish to show that u vanishes. Let $F \in \Delta_{n-1}(T)$ be a facet of T, and consider $\operatorname{tr}_F u$, which is a polynomial function of at most degree r on the $(n-1)$-dimensional simplex F, i.e., it belongs to $\mathcal{P}_r(F)$. Moreover, if we replace T by F and u by $\operatorname{tr}_F u$ in (2.1), the resulting functionals vanish by assumption (using the obvious fact that $\operatorname{tr}_f \operatorname{tr}_F u = \operatorname{tr}_f u$ for $f \subset F \subset T$). By induction we conclude that $\operatorname{tr}_F u$ vanishes on all the facets F of T. Therefore, u is divisible by the barycentric coordinate function λ_i which vanishes on F, and, since this holds for all facets, $u = (\prod_{i=0}^{n} \lambda_i) p$ for some $p \in \mathcal{P}_{r-n-1}(T)$. Taking $f = T$ and $q = p$ in (2.1) we conclude that

$$\int_T \left(\prod_{i=0}^{n} \lambda_i \right) p^2 = 0,$$

which implies that p vanishes on T, and so u does as well. $\square$

Let us note some features of the proof, which will be common to the unisolvence proofs for all of the finite element spaces we discuss here. After a dimension count to verify that the proposed degrees of freedom are correct in number, or at least no more than required, the proof proceeded by induction on the number of space dimensions. The inductive step relied on a trace property of the shape function space $V(T) = \mathcal{P}_r(T)$ for the family, namely that $\operatorname{tr}_F V(T) \subset V(F)$. Moreover, it used a similar trace property for the degrees of freedom: if $\xi_F \in V(F)^*$ is a degree of freedom for $V(F)$, then the pullback $\xi_F \circ \operatorname{tr}_F \in V(T)^*$ is a degree of freedom for $V(T)$. The induction reduced the unisolvence proof to verifying that if $u \in \mathring{V}(T)$, the space of functions in $V(T)$ whose trace vanishes on the entire boundary, and if the interior degrees of freedom (those associated to T itself) of u vanish, then u itself vanishes, which we showed by explicit construction.

Finally, we note that the continuity implied by the degrees of freedom is exactly what is required to insure that the Lagrange finite element space is contained in H^1:

$$\mathcal{P}_r \Lambda^0(\mathcal{T}_h) = \{ u \in H^1(\Omega) \mid u \text{ belongs to } \mathcal{P}_r(T) \text{ piecewise} \}. \tag{2.2}$$

Indeed, a piecewise smooth function belongs to $H^1(\Omega)$ if and only if its traces on faces are single-valued. Thus if a function in $H^1(\Omega)$ belongs piecewise to $\mathcal{P}_r(T)$, its traces are single-valued, so the degrees of freedom are single-valued, and the function belongs to $\mathcal{P}_r \Lambda^0(\mathcal{T}_h)$. On the other hand, if the function belongs to $\mathcal{P}_r \Lambda^0(\mathcal{T}_h)$, its traces on faces are single-valued, since, as we saw in the course of the unisolvence proof, they are determined by the degrees of freedom. Thus the function belongs to $H^1(\Omega)$.

2.3 Exterior Calculus

For the convenience of readers less familiar with differential forms and exterior calculus we now briefly review key definitions and properties. We begin with the space of *algebraic* k-forms on V: $\operatorname{Alt}^k V = \{L : V^k \to \mathbb{R} \mid k\text{-linear, skew-symmetric}\}$, where the multilinear form L is skew-symmetric, or alternating, if it changes sign under the interchange of any two of its arguments. The skew-symmetry condition is vacuous if $k < 2$, so $\operatorname{Alt}^1 V = V^*$ and, by convention, $\operatorname{Alt}^0 V = \mathbb{R}$. If ω is any k-linear map $V^k \to \mathbb{R}$, then $\operatorname{skw} \omega \in \operatorname{Alt}^k V$ where

$$(\operatorname{skw} \omega)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma) \omega(v_{\sigma_1}, \ldots, v_{\sigma_k}),$$

with the sum taken over all the permutations of the integers 1 to k. The wedge product $\mathrm{Alt}^k V \times \mathrm{Alt}^l V \to \mathrm{Alt}^{k+l} V$ is defined

$$\omega \wedge \mu = \binom{k+l}{k} \operatorname{skw}(\omega \otimes \mu), \quad \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^l V.$$

Let $v_1, \ldots, v_n$ form a basis for V. Denoting by

$$\Sigma(k,n) = \{(\sigma_1, \ldots, \sigma_k) \in \mathbb{N}^k \mid 1 \le \sigma_1 < \cdots < \sigma_k \le n\},$$

an element of $\mathrm{Alt}^k V$ is completely determined by the values it assigns to the k-tuples $(v_{\sigma_1}, \ldots, v_{\sigma_k})$, $\sigma \in \Sigma_k$. Moreover, these values can be assigned arbitrarily. In fact, the k-form $\mu_{\sigma_1} \wedge \cdots \wedge \mu_{\sigma_k}$, where $\mu_1, \ldots, \mu_n$ is the dual basis to $v_1, \ldots, v_n$, takes the k-tuple $(v_{\sigma_1}, \ldots, v_{\sigma_k})$ to 1, and the other such k-tuples to 0. Thus $\dim \mathrm{Alt}^k V = \binom{n}{k}$, where $n = \dim V$.

We define differential forms on an arbitrary manifold, since we will be using them both when the manifold is a domain in $\mathbb{R}^n$ and when it is the boundary of such a domain. A differential k-form on a manifold Ω is a map ω which takes each point $x \in \Omega$ to an element $\omega_x \in \mathrm{Alt}^k T_x\Omega$, where $T_x\Omega$ is the tangent space to Ω at x. In other language, ω is a skew-symmetric covariant tensor field on Ω of order k. In particular, a differential 0-form is just a real-valued function on Ω and a differential 1-form is a covector field. In the case Ω is a domain in $\mathbb{R}^n$, then each tangent space can be identified with $\mathbb{R}^n$, and a differential k-form is simply a map $\Omega \to \mathrm{Alt}^k \mathbb{R}^n$. In this context, it is common to denote the dual basis to the canonical basis for $\mathbb{R}^n$ by $dx^1, \ldots, dx^n$, so dx^k applied to a vector $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ is its kth component v^k. With this notation, an arbitrary differential k-form can be written

$$u(x) = \sum_{\sigma \in \Sigma(k,n)} a_\sigma(x)\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k},$$

for some coefficients $a_\sigma : \Omega \to \mathbb{R}$.

Three basic operations on differential forms are the exterior derivative, the form integral, and the pullback. The exterior derivative $d\omega$ of a k-form ω is a $(k+1)$-form. In the case of a domain in $\mathbb{R}^n$, it is given by the intuitive formula

$$d\big(a_\sigma\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}\big) = \sum_{j=1}^{n} \frac{\partial a_\sigma}{\partial x_j}\, dx^j \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}.$$

It satisfies (in general) the identity $d^{k+1} \circ d^k = 0$ and the Leibniz rule $d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^k \omega \wedge (d\mu)$ if ω is a k-form.

The definition of the form integral requires that the manifold Ω be oriented. In this case we can define $\int_\Omega \omega \in \mathbb{R}$ for ω an n-form with $n = \dim \Omega$. The integral changes sign if the orientation of the manifold is reversed.

Finally, if $F : \Omega \to \Omega'$ is a differentiable map, then the pullback F^* takes a k-form on Ω' to one on Ω by

$$\big(F^*\omega\big)_x(v_1, \ldots, v_k) = \omega_{F(x)}(dF_x v_1, \ldots, dF_x v_k), \quad x \in \Omega,\ v_1, \ldots, v_k \in T_x\Omega.$$

The pullback respects the operations of wedge product, exterior derivative, and form integral:

$$F^*(\omega \wedge \mu) = \big(F^*\omega\big) \wedge \big(F^*\mu\big), \quad F^*(d\omega) = d\big(F^*\omega\big), \quad \int_\Omega F^*\omega = \int_{\Omega'} \omega,$$

for ω and μ differential forms on Ω'. The last relation requires that F be a diffeomorphism of Ω with Ω' which preserves orientation.

An important special case of pullback is when F is the inclusion of a submanifold Ω into a larger manifold Ω'. In this case the pullback is the trace operator taking a k-form on Ω' to a k-form on the submanifold Ω. All these operations combine elegantly into Stokes' theorem, which says that, under minimal hypothesis on the smoothness of the differential $(n-1)$-form ω and the n-manifold Ω,

$$\int_{\partial\Omega} \operatorname{tr}\omega = \int_\Omega d\omega.$$

If V is an inner product space, then there is a natural inner product on $\operatorname{Alt}^k V$. Thus for a Riemannian manifold, such as any manifold embedded in $\mathbb{R}^n$, the inner product $\langle \omega_x, \mu_x \rangle \in \mathbb{R}$ is defined for any k-forms ω, μ and any $x \in \Omega$. An oriented Riemannian manifold also has a unique volume form, vol, a differential n-form which at each point assigns the value 1 to a positively oriented orthonormal basis for the tangent space at that point. For a subdomain of $\mathbb{R}^n$ the volume form is the constant n-form with the value $dx^1 \wedge \cdots \wedge dx^n$ at each point. Combining these notions, we see that on any oriented Riemannian manifold we may define the L^2-inner product of k-forms:

$$\langle \omega, \mu \rangle_{L^2\Lambda^k(\Omega)} = \int_\Omega \langle \omega_x, \mu_x \rangle \, \mathsf{vol}.$$

The space $L^2\Lambda^k$ is of course the space of k-forms for which $\|\omega\|_{L^2\Lambda^k} := \sqrt{\langle \omega, \omega \rangle_{L^2\Lambda^k}} < \infty$, and then $H\Lambda^k$ is defined as in (1.1).

3 Families of Finite Element Differential Forms on Simplicial Meshes

Our goal now is to create finite element subspaces of the spaces $H\Lambda^k$ which fit together to yield a subcomplex with commuting projections. In this section the spaces will be constructed for a simplicial triangulation $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^n$. Thus, for a simplex T, we must specify a space $V(T)$ of polynomial differential forms and a set of degrees of freedom for it.

3.1 The Polynomial Space $\mathcal{P}_r \Lambda^k$

An obvious choice for $V(T)$ is the space

$$\mathcal{P}_r \Lambda^k(T) = \Big\{ \sum_{\sigma \in \Sigma(k,n)} p_\sigma \, dx^\sigma \;\Big|\; p_\sigma \in \mathcal{P}_r(T) \Big\},$$

of a differential k-forms with polynomial coefficients of degree at most r. It is easy to compute its dimension:

$$\dim \mathcal{P}_r \Lambda^k(T) = \#\Sigma(k,n) \times \dim \mathcal{P}_r(T) = \binom{n}{k}\binom{n+r}{n} = \binom{n+r}{n-k}\binom{r+k}{r}. \tag{3.1}$$

Note that $d\mathcal{P}_r \Lambda^k \subset \mathcal{P}_{r-1}\Lambda^{k+1}$, i.e., the exterior derivative lowers the polynomial degree at the same time as it raises the form degree. Therefore, for each r we have a subcomplex of the de Rham complex:

$$\mathcal{P}_r \Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0. \tag{3.2}$$

This complex is exact (we have left off the initial 0 since the first map, $d = \text{grad}$ acting on $\mathcal{P}_r \Lambda^0$ has a 1-dimensional kernel, consisting of the constant functions). That is, if $\omega \in \mathcal{P}_s \Lambda^k$ and $d\omega = 0$ then $\omega = d\mu$ for some $\mu \in \mathcal{P}_{s+1}\Lambda^{k-1}$. We prove this in Corollary 3.2 below, using an elementary but powerful tool called the *Koszul complex*. The same tool will also be used to define the degrees of freedom for $\mathcal{P}_r \Lambda^k(T)$, and to define an alternative space of shape functions.

3.2 The Koszul Complex

For a domain in $\Omega \subset \mathbb{R}^n$ (but not a general manifold), the identity map may be viewed as a vector field. It assigns to an arbitrary point $x \in \Omega \subset \mathbb{R}^n$ the point itself viewed as a vector in $\mathbb{R}^n$ and so an element of the tangent space $T_x\Omega$. Contracting a k-form ω with this identity vector field gives a $(k-1)$-form $\kappa\omega$:

$$(\kappa\omega)_x(v_1, \dots, v_{k-1}) = \omega_x(x, v_1, \dots, v_{k-1}), \quad x \in \Omega, \; v_1, \dots, v_{k-1} \in \mathbb{R}^n.$$

Since ω_x is skew-symmetric, $\kappa\kappa\omega = 0$, that is, κ is a differential. It satisfies a Leibniz rule:

$$\kappa(\omega \wedge \mu) = (\kappa\omega) \wedge \mu + (-1)^k \omega \wedge (\kappa\mu),$$

for a k-form ω and a second form μ. In particular $\kappa(f\omega) = f\kappa\omega$ if f is a function.

Also $\kappa dx^i = x^i$. These properties fully determine κ. Thus

$$\kappa\left(dx^i \wedge dx^j\right) = x^i\, dx^j - x^j\, dx^i,$$
$$\kappa\left(dx^i \wedge dx^j \wedge dx^k\right) = x^i\, dx^j \wedge dx^k - x^j\, dx^i \wedge dx^k + x^k\, dx^i \wedge dx^j,$$

and so forth. If we identify 1-forms with vector fields, then κ corresponds to the dot product of the vector field with x (or, more properly, with the identity vector field). On 2-forms in 3-D, κ is the cross product with x, and on 3-forms it is the product of a scalar field with x to get a vector field.

The Koszul differential κ maps the space $\mathcal{P}_r\Lambda^k$ of differential k-forms with coefficients in $\mathcal{P}_r(\Omega)$ to $\mathcal{P}_{r+1}\Lambda^{k-1}$, exactly the reverse of d. Thus both κd and $d\kappa$ map $\mathcal{P}_r\Lambda^k$ to itself. The following theorem points to an intimate relation between κ and d, called the *homotopy formula*. In it we write $\mathcal{H}_r\Lambda^k$ for the k-forms with *homogeneous* polynomial coefficients of degree r.

Theorem 3.1 (Homotopy formula)

$$(\kappa d + d\kappa)\omega = (k+r)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k.$$

Remarks on the proof The case $k=0$ is Euler's identity $x \cdot \operatorname{grad} p = r\,p$ for p a homogeneous polynomial of degree r. Using it, we can verify the theorem by direct computation. Alternatively, one may use Cartan's homotopy formula from differential geometry. For details on both proofs, see Theorem 3.1 of [5]. □

Corollary 3.2 *The polynomial de Rham complex* (3.2) *and the Koszul complex*

$$0 \to \mathcal{P}_{r-n}\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{P}_r\Lambda^0$$

are both exact.

Proof For the de Rham complex, it suffices to establish exactness of the homogeneous polynomial de Rham complex

$$\mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \to 0,$$

since then we can then just sum to get the result. We must show that if $\omega \in \mathcal{H}_s\Lambda^k$ and $d\omega = 0$ then ω is in the range of d. Indeed, by the homotopy formula

$$\omega = (s+k)^{-1}(d\kappa + \kappa d)\omega = (s+k)^{-1} d\kappa\omega.$$

A similar proof holds for the Koszul complex. □

Another important consequence is a direct sum decomposition:

Corollary 3.3 *For $r \ge 1$, $0 \le k \le n$,*

$$\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}. \tag{3.3}$$

Proof By the homotopy formula, any element of $\mathcal{H}_r\Lambda^k$ belongs to $\kappa\mathcal{H}_{r-1}\Lambda^{k+1} + d\mathcal{H}_{r+1}\Lambda^{k-1}$. Moreover the intersection of these two spaces is zero, since if ω belongs to the intersection, then $d\omega = 0$, $\kappa\omega = 0$, so $\omega = 0$ by the homotopy formula. □

3.3 The Polynomial Space $\mathcal{P}_r^-\Lambda^k$

We now define a second space of polynomial differential forms which can be used as shape functions. We have

$$\mathcal{P}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \mathcal{H}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}.$$

If we drop the last summand, we get a space intermediate between $\mathcal{P}_{r-1}\Lambda^k$ and $\mathcal{P}_r\Lambda^k$:

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}. \tag{3.4}$$

Note that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$, but for $0 < k < n$, $\mathcal{P}_r^-\Lambda^k$ is contained strictly between $\mathcal{P}_{r-1}\Lambda^k$ and $\mathcal{P}_r\Lambda^k$. We may compute the dimension of $\kappa\mathcal{H}_r\Lambda^k$, using the exactness of the Koszul complex and induction (see [5, Theorem 3.3]). This then yields a formula for the dimension of $\mathcal{P}_r^-\Lambda^k$:

$$\dim \mathcal{P}_r^-\Lambda^k = \binom{n+r}{n-k}\binom{r+k-1}{k}.$$

Comparing this with (3.1), we have

$$\dim \mathcal{P}_r^-\Lambda^k = \frac{r}{r+k}\dim \mathcal{P}_r\Lambda^k$$

(showing again that the spaces coincide for 0-forms).

Now

$$d\mathcal{P}_r^-\Lambda^k \subset d\mathcal{P}_r\Lambda^k \subset \mathcal{P}_{r-1}\Lambda^{k+1} \subset \mathcal{P}_r^-\Lambda^{k+1},$$

so we obtain another subcomplex of the de Rham complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0. \tag{3.5}$$

Note that, in contrast to (3.2), in this complex the degree r is held constant. However, like (3.2), the complex (3.5) is exact. Indeed,

$$\begin{aligned} d\mathcal{P}_r^-\Lambda^k &= d\big(\mathcal{P}_r^-\Lambda^k + d\mathcal{P}_{r+1}\Lambda^{k-1}\big) = d\mathcal{P}_r\Lambda^k \\ &= \mathcal{N}\big(d|\mathcal{P}_{r-1}\Lambda^{k+1}\big) = \mathcal{N}\big(d|\mathcal{P}_r^-\Lambda^{k+1}\big), \end{aligned}$$

where the penultimate equality follows from Corollary 3.2 and the last equality is a consequence of the definition (3.4) and the homotopy formula Theorem 3.1.

3.4 The $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ Family of Finite Element Differential Forms

Let $r \ge 1$, $0 \le k \le n$, and let $\mathcal{T}_h$ be a simplicial mesh of $\Omega \subset \mathbb{R}^n$. We define a finite element subspace $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ of $H\Lambda^k(\Omega)$. As shape functions on a simplex $T \in \mathcal{T}_h$ we take $V(T) = \mathcal{P}_r^- \Lambda^k(T)$. As degrees of freedom we take

$$u \in \mathcal{P}_r^- \Lambda^k(T) \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta_d(T),\ d \ge k. \tag{3.6}$$

Note that, in the case $k = 0$, $V(T) = \mathcal{P}_r(T)$ and (3.6) coincides with (2.1), so the space $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ generalizes the Lagrange finite elements to differential forms of arbitrary form degree. We shall prove unisolvence for arbitrary polynomial degree, form degree, and space dimension at once. The proof will use the following lemma, which is proved via a simple construction using barycentric coordinates.

Lemma 3.4 *Let $r \ge 1$, $0 \le k \le n$, and let T be an n-simplex. If $u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$ and*

$$\int_T u \wedge q = 0, \quad q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T), \tag{3.7}$$

then $u \equiv 0$.

Proof Any element of $\mathcal{P}_{r-1}\Lambda^k(T)$ can be written in terms of barycentric coordinates as

$$u = \sum_{\sigma \in \Sigma(k,n)} u_\sigma d\lambda_{\sigma_1} \wedge \cdots \wedge d\lambda_{\sigma_k}, \quad u_\sigma \in \mathcal{P}_{r-1}(T).$$

Now let $1 \le i \le n$, and consider the trace of u on the face given by $\lambda_i = 0$. By the assumption that $u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$, the trace vanishes. This implies that λ_i divides u_σ for any $\sigma \in \Sigma(k,n)$ whose range does not contain i. Thus

$$u_\sigma = p_\sigma \lambda_{\sigma_1^*} \cdots \lambda_{\sigma_{n-k}^*} \quad \text{for some } p_\sigma \in \mathcal{P}_{r+k-n-1}(T),$$

where $\sigma^* \in \Sigma(n-k,n)$ is the increasing sequence complementary to σ. Thus

$$u = \sum_{\sigma \in \Sigma(k,n)} p_\sigma \lambda_{\sigma_1^*} \cdots \lambda_{\sigma_{n-k}^*} d\lambda_{\sigma_1} \wedge \cdots \wedge d\lambda_{\sigma_k}, \quad p_\sigma \in \mathcal{P}_{r+k-n-1}(T).$$

Choosing

$$q = \sum_{\sigma \in \Sigma(k,n)} (-1)^{\operatorname{sign}(\sigma,\sigma^*)} p_\sigma d\lambda_{\sigma_1^*} \wedge \cdots \wedge d\lambda_{\sigma_{n-k}^*}$$

in (3.7), we get

$$0 = \int_T u \wedge q = \int_T \sum_{\sigma \in \Sigma(k,n)} p_\sigma^2 \lambda_{\sigma_1^*} \cdots \lambda_{\sigma_{n-k}^*} d\lambda_1 \wedge \cdots \wedge d\lambda_n.$$

However, the λ_i are positive on the interior of T and the n-form $d\lambda_1 \wedge \cdots \wedge d\lambda_n$ is a nonzero multiple of the volume form. Thus each p_σ must vanish, and so u vanishes. □

Theorem 3.5 (Unisolvence for $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$) *For any $r \geq 1$, $0 \leq k \leq n$, and n-simplex T, the degrees of freedom* (3.6) *are unisolvent for $V(T) = \mathcal{P}_r^- \Lambda^k(T)$.*

Proof First we do the dimension count. The number of degrees of freedom is at most

$$\sum_{d \geq k} \#\Delta_d(T) \dim \mathcal{P}_{r+k-d-1} \Lambda^k(\mathbb{R}^d) = \sum_{d \geq k} \binom{n+1}{d+1} \binom{r+k-1}{d} \binom{d}{k}$$
$$= \sum_{j \geq 0} \binom{n+1}{j+k+1} \binom{r+k-1}{j+k} \binom{j+k}{j}.$$

Simplifying with the binomial identities,

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{a-b}, \quad \sum_{j \geq 0} \binom{a}{b+j}\binom{c}{j} = \binom{a+c}{a-b},$$

the right-hand side becomes

$$\binom{r+n}{r+k}\binom{r+k-1}{k} = \dim \mathcal{P}_r^- \Lambda^k(T).$$

It remains to show that if $u \in \mathcal{P}_r^- \Lambda^k(T)$ and the degrees of freedom in (3.6) vanish, then u vanishes. Since $\operatorname{tr}_f \mathcal{P}_r^- \Lambda^k(T) = \mathcal{P}_r^- \Lambda^k(f)$, we may use induction on dimension to conclude that $\operatorname{tr}_f u$ vanishes on each facet f, so $u \in \mathring{\mathcal{P}}_r^- \Lambda^k(T)$. Therefore $du \in \mathring{\mathcal{P}}_{r-1} \Lambda^{k+1}(T)$. Moreover,

$$\int_T du \wedge p = \pm \int_T u \wedge dp = 0, \quad p \in \mathcal{P}_{r+k-n} \Lambda^{n-k-1}(T),$$

where the first equality comes from Stoke's theorem and the Leibniz rule, and the second from the hypothesis that the degrees of freedom for u vanish. We may now apply the lemma (with k replaced by $k+1$) to du to conclude that du vanishes. But the homotopy formula implies that for $u \in \mathcal{P}_r^- \Lambda^k$ with $du = 0$, $u \in \mathcal{P}_{r-1} \Lambda^k$. Using the interior degrees of freedom from (3.6), we may apply the lemma to u, to conclude that u vanishes. □

It is easy to check that the degrees of freedom imply single-valuedness of the traces of elements of $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$, so that they indeed belong to $H\Lambda^k$. Moreover, it is easy to see that the complex (3.5) involving the shape functions, leads to a finite element subcomplex of the L^2 de Rham complex on Ω:

$$\mathcal{P}_r^- \Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r^- \Lambda^1 \xrightarrow{d} (\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^- \Lambda^n(\mathcal{T}_h).$$

Using the degrees of freedom to define projection operators π_h^k into $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ (the domain of π_h^k consists of all continuous k-forms in $H\Lambda^k(\Omega)$), we obtain projections that commute with d (this can be verified using Stokes' theorem), which is crucial to the analysis of the element via FEEC.

3.5 The $\mathcal{P}_r \Lambda^k(\mathcal{T}_h)$ Family of Finite Element Differential Forms

We may also use the full polynomial space $\mathcal{P}_r \Lambda^k(T)$ as shape functions for a finite element space. The corresponding degrees of freedom are

$$u \in \mathcal{P}_r \Lambda^k(T) \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d}^- \Lambda^{d-k}(f), \; f \in \Delta_d(T), \; d \geq k. \tag{3.8}$$

Note that in this case the degrees of freedom involve $\mathcal{P}_r^-$ spaces, defined through the Koszul complex. The analysis of these spaces is very parallel to that of the last subsection, and we will not carry it out here. Again, we obtain unisolvence, and a finite element subcomplex of the de Rham complex

$$\mathcal{P}_r \Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1} \Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n} \Lambda^n(\mathcal{T}_h),$$

which admits a commuting projection defined via the degrees of freedom.

3.6 Historical Notes

In the case $k = 0$, the two shape function spaces $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ coincide, as do the spaces $\mathcal{P}_{r-d-1} \Lambda^{d-k}(f)$ and $\mathcal{P}_{r-d}^- \Lambda^{d-k}(f)$, $f \in \Delta_d(T)$, entering (3.6) and (3.8). Thus the two finite element families coincide for 0-forms, and provide two distinct generalizations of the Lagrange elements to differential forms of higher degree.

In n dimensions, n-forms may be viewed as scalar functions and the space $H\Lambda^n(\Omega)$ just corresponds to $L^2(\Omega)$. The finite element subspace $\mathcal{P}_r \Lambda^n(\mathcal{T}_h)$ is simply the space of all piecewise polynomial functions of degree r, with no interelement continuity required. The space $\mathcal{P}_r^- \Lambda^n(\mathcal{T}_h)$ coincides with $\mathcal{P}_{r-1} \Lambda^n(\mathcal{T}_h)$.

In two dimensions, the remaining spaces $\mathcal{P}_r^- \Lambda^1(\mathcal{T}_h)$ and $\mathcal{P}_r \Lambda^1(\mathcal{T}_h)$ can be identified, via vector proxies, with the Raviart–Thomas spaces [20] and the Brezzi–Douglas–Marini spaces [10]. In three dimensions, the $\mathcal{P}_r^- \Lambda^1(\mathcal{T}_h)$ and $\mathcal{P}_r^- \Lambda^2(\mathcal{T}_h)$ spaces are the finite element subspaces of $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$, respectively, called the Nédélec edge and face elements of the first kind [18]. The spaces $\mathcal{P}_r \Lambda^1(\mathcal{T}_h)$ and $\mathcal{P}_r \Lambda^2(\mathcal{T}_h)$ are the Nédélec edge and face elements of the second kind [19]. Diagrams for the two-dimensional and three-dimensional elements are shown in Figs. 2 and 3.

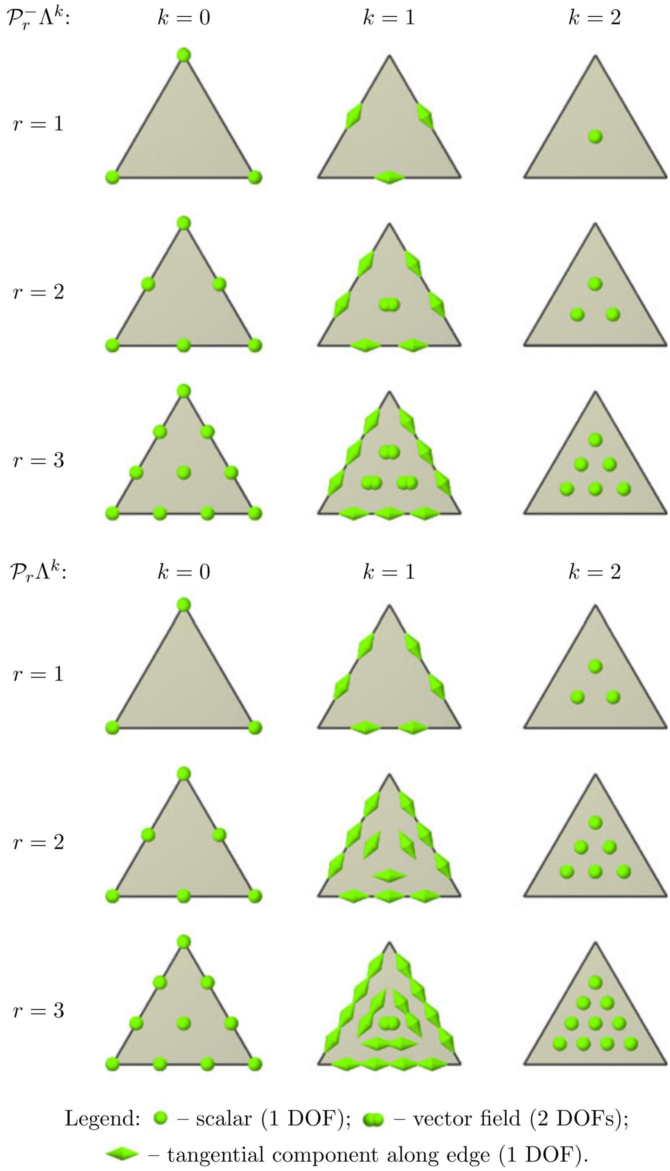


Fig. 2 The $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ spaces in two dimensions

The lowest order spaces $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ are very geometric, possessing precisely one degree of freedom per face of dimension k, and no others (see the top rows of Figs. 2 and 3). In fact these spaces first appeared in the geometry literature in the work of Whitney in 1957 [24] long before their first appearance as finite elements. In the 1970s, they were used by Dodziuk [13] and Dodziuk and Patodi [14] as a theoret-

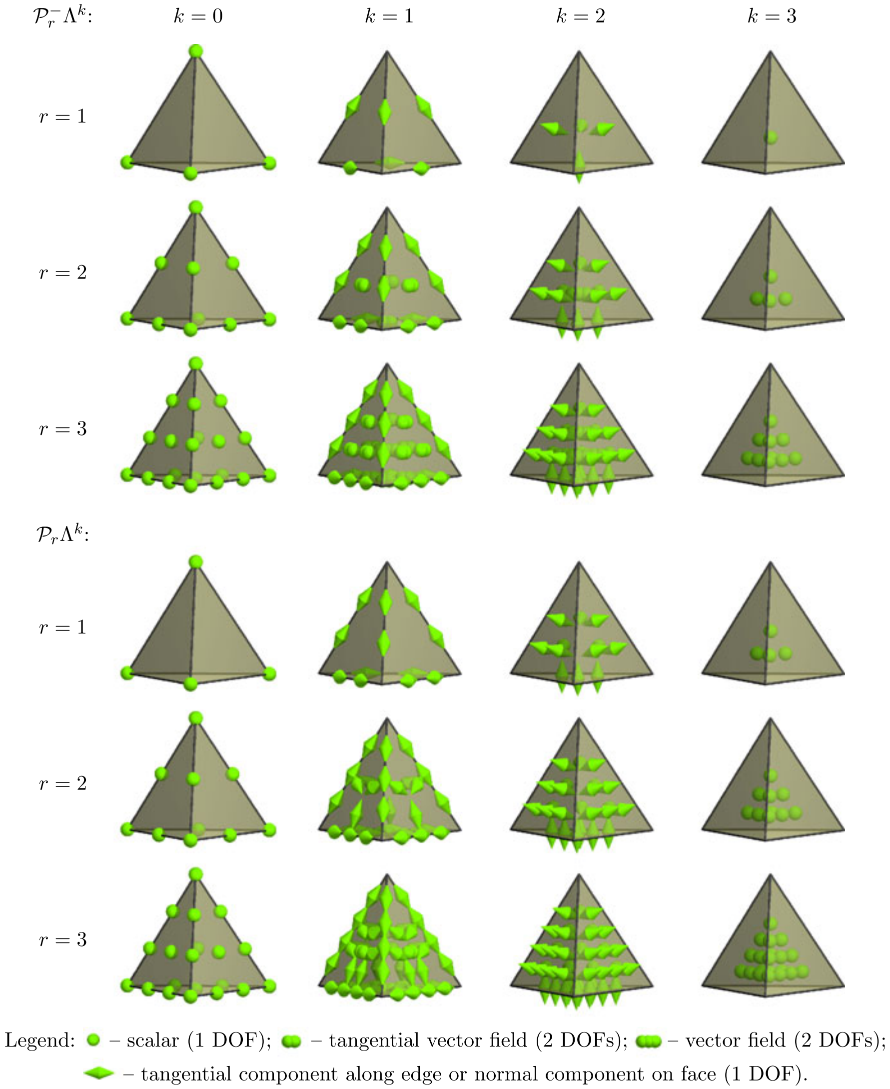


Fig. 3 The $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r \Lambda^k(\mathcal{T}_h)$ spaces in three dimensions

ical tool to approximate the eigenvalues of the Hodge Laplacian on a Riemannian manifold. This then played an essential role in Müller's proof of the Ray–Singer conjecture [17]. The spaces $\mathcal{P}_r \Lambda^k(\mathcal{T}_h)$ also appeared in the geometry literature, introduced by Sullivan [22, 23]. In an early, largely overlooked paper bringing finite element analysis techniques to bear on geometry Baker [7] named these Sullivan–Whitney forms, and analyzed their convergence for the eigenvalue problem for the Hodge Laplacian. In 1988 Bossavit made the connection between Whitney forms and the mixed finite elements in use in electromagnetics [8], in part inspired by the thesis of Kotiuga [16]. The first unified treatment of the $\mathcal{P}_r^- \Lambda^k$ spaces, which was based on exterior calculus and included a unisolvence proof, was in a seminal paper of Hiptmair [15] in 1999. In the 2006 paper of Arnold, Falk, and Winther [5], in which the term finite element exterior calculus first appeared, the Koszul complex was first applied to finite elements, simplifying many aspects and resulting in a simultaneous treatment of both the $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ spaces.

4 Families of Finite Element Differential Forms on Cubical Meshes

We now describe two families of spaces of finite element differential forms, which we denote $\mathcal{Q}_r^- \Lambda^k(\mathcal{T}_h)$ and $\mathcal{S}_r \Lambda^k(\mathcal{T}_h)$, defined for cubical meshes $\mathcal{T}_h$, i.e., meshes in which each element is the Cartesian product of intervals. In some sense, the $\mathcal{Q}_r^- \Lambda^k$ family can be seen as an analogue of the $\mathcal{P}_r^- \Lambda^k$ family for simplicial meshes, and the $\mathcal{S}_r \Lambda^k$ family an analogue of the $\mathcal{P}_r \Lambda^k$ family. The $\mathcal{Q}_r^- \Lambda^k$ family can be constructed from the one-dimensional case by a tensor product construction, and is long known. By contrast, the $\mathcal{S}_r \Lambda^k$ family first appeared in recent work of Arnold and Awanou [3]. Even in two and three dimensions, the spaces in this family were for the most part not known previously.

4.1 The $\mathcal{Q}_r^- \Lambda^k$ Family

We describe this family only very briefly. A more detailed description will be included in a forthcoming study of the approximation properties of these spaces under non-affine mappings [4]. Suppose we are given a subcomplex of the de Rham complex on an element $S \subset \mathbb{R}^m$ and a second such subcomplex on an element $T \subset \mathbb{R}^n$:

$$V^0(S) \xrightarrow{d} V^1(S) \xrightarrow{d} \cdots \xrightarrow{d} V^m(S), \qquad V^0(T) \xrightarrow{d} V^1(T) \xrightarrow{d} \cdots \xrightarrow{d} V^n(T).$$

We may then construct a subcomplex of the de Rham complex on $S \times T$ by a tensor product construction which is known in the theory of differential forms; see, e.g., [21, p. 61]. The canonical projection $\pi_S : S \times T \to S$ determines a pullback of i-forms on S to i-forms on $S \times T$, so $\pi_S^* V^i(S)$ is a space of i-forms on $S \times T$ and, similarly, $\pi_T^* V^j(T)$ is a space of j-forms on $S \times T$. Thus we may define a space of k-forms on $S \times T$ by

$$V^k(S \times T) = \bigoplus_{i+j=k} \pi_S^* V^i(S) \wedge \pi_T^* V^j(T).$$

We take the space $V^k(S \times T)$ as the shape functions for k-forms on $S \times T$. The construction of degrees of freedom for $V^k(S \times T)$ is simple. If $\eta \in V^i(S)^*$ is a degree of freedom associated to a face f of S, and $\rho \in V^j(T)^*$ is associated to a face g of T, we define

$$\eta \wedge \rho \in \left[\pi_S^* V^i(S) \wedge \pi_T^* V^j(T)\right]^* \subset V^k(S \times T)^*,$$

by

$$(\eta \wedge \rho)\big(\pi_s^* u \wedge \pi_T^* v\big) = \eta(u)\rho(v),$$

and associate the degree of freedom $\eta \wedge \rho$ to $f \times g$, which is a face of $S \times T$.

The $\mathcal{Q}_r^-$ family is defined by applying this tensor product repeatedly, starting with a finite element de Rham complex on an interval in one dimension. In one dimension the $\mathcal{P}_r^-$ and $\mathcal{P}_r$ de Rham subcomplexes coincide. On an interval I, the shape functions for 0-forms are $V^0(I) = \mathcal{P}_r(I)$ with degrees of freedom at each end point, and moments of degree at most $r-1$ in the interior. The shape function for 1-forms are $V^1(I) = \mathcal{P}_{r-1}(I)$ with all degrees of freedom in the interior. Repeatedly using the tensor product construction just outlined, we obtain polynomial spaces and degrees of freedom on a box $I_1 \times \cdots \times I_n \subset \mathbb{R}^n$. We denote the shape function space so obtained by $\mathcal{Q}_r^- \Lambda^k(I_1 \times \cdots \times I_n)$. In $n = 2$ dimensions, for example,

$$\mathcal{Q}_r^- \Lambda^0(I_1 \times I_2) = \mathcal{Q}_r(I_1 \times I_2) = \mathcal{P}_r(I_1) \otimes \mathcal{P}_r(I_2),$$

$$\mathcal{Q}_r^- \Lambda^1(I_1 \times I_2) = \left[\mathcal{P}_{r-1}(I_1) \otimes \mathcal{P}_r(I_2)\right] \times \left[\mathcal{P}_r(I_1) \otimes \mathcal{P}_{r-1}(I_2)\right],$$

$$\mathcal{Q}_r^- \Lambda^2(I_1 \times I_2) = \mathcal{Q}_{r-1}(I_1 \times I_2).$$

Diagrams for these elements in two and three dimensions are shown in Fig. 4. The space $\mathcal{Q}_r^- \Lambda^0(\mathcal{T}_h)$ is the standard $\mathcal{Q}_r$ finite element subspace of $H^1(\Omega)$ and the space $\mathcal{Q}_r^- \Lambda^n(\mathcal{T}_h)$ is the discontinuous $\mathcal{Q}_{r-1}$ subspace of $L^2(\Omega)$. The space $\mathcal{Q}_r^- \Lambda^1(\mathcal{T}_h)$ goes back to Raviart and Thomas [20] in two dimensions, and the $\mathcal{Q}_r^- \Lambda^1(\mathcal{T}_h)$ and $\mathcal{Q}_r^- \Lambda^2(\mathcal{T}_h)$ were given by Nédélec in [18]. The spaces with r held fixed combine to create a finite element de Rham subcomplex,

$$\mathcal{Q}_r^- \Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{Q}_r^- \Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^- \Lambda^n(\mathcal{T}_h),$$

and the degrees of freedom determine commuting projections.

Recently, Cockburn and Qiu [12] have published a different family of finite element spaces in two and three dimensions, that seems to be related to these. They begin with the complex formed by the full spaces $\mathcal{Q}_r \Lambda^k$, which lie between $\mathcal{Q}_r^- \Lambda^k$ and $\mathcal{Q}_{r+1}^- \Lambda^k$. That complex (which was discussed in [19]) does *not* admit commuting projections. Cockburn and Qiu define a small space of bubble functions that can be added to each of the spaces so that the resulting spaces remain inside $\mathcal{Q}_{r+1}^- \Lambda^k$ but also form a de Rham subcomplex (with constant r) which admits commuting projections.

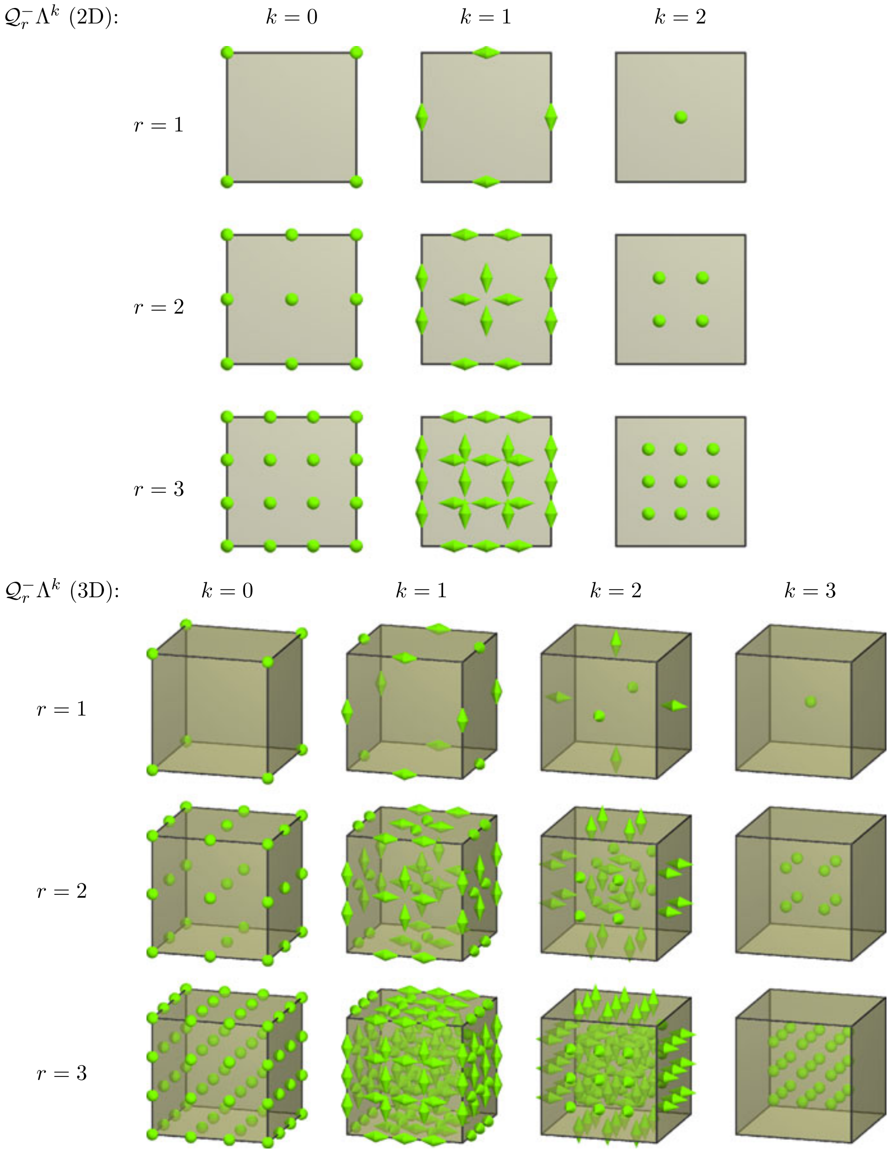


Fig. 4 The $\mathcal{Q}_r^- \Lambda^k(\mathcal{T}_h)$ spaces in two and three dimensions

4.2 A Second Family of Finite Element Differential Forms on Cubes

The $\mathcal{S}_r \Lambda^k$ family presented in this section was derived recently in [3]. It seems to be complementary to the $\mathcal{Q}_r^- \Lambda^k$ family much as the $\mathcal{P}_r \Lambda^k$ family complements the $\mathcal{P}_r^- \Lambda^k$ family. To describe the new family we require some notation. A k-form monomial in n variables is the product of an ordinary monomial and a simple alternator:

$$m = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k},$$

where α is a multi-index and $\sigma \in \Sigma(k,n)$. We define the degree of m to be the polynomial degree of its coefficient: $\deg m = \sum_i \alpha_i$. The *linear degree* of m is more complicated:

$$\operatorname{ldeg} m = \#\{i \mid \alpha_i = 1,\ \alpha_i \notin \{\sigma_1, \dots, \sigma_k\}\},$$

that is, the number of variables that enter the coefficient linearly, not counting the variables that enter the alternator. For example, if $m = x^1 x^2 (x^3)^5\, dx^1$, then $\deg m = 7$, $\operatorname{ldeg} m = 1$.

We now define the space of shape functions we shall use for k-forms on an n-dimensional box, T. Viewing monomial forms as differential forms on T, we define $\mathcal{H}_{r,l}\Lambda^k(T) \subset \mathcal{H}_r\Lambda^k(T)$ to be the span of all monomial k-forms m such that $\deg m = r$ and $\operatorname{ldeg} m \geq l$. Using this definition and the Koszul differential, we then define

$$\mathcal{J}_r\Lambda^k(T) = \sum_{l \geq 1} \kappa \mathcal{H}_{r+l-1,l}\Lambda^{k+1}(T) \subset \mathcal{P}_{r+n-k-1}\Lambda^k(T).$$

Finally, we define the shape functions on T by

$$\mathcal{S}_r\Lambda^k(T) = \mathcal{P}_r\Lambda^k(T) + \mathcal{J}_r\Lambda^k(T) + d\mathcal{J}_{r+1}\Lambda^{k-1}(T),$$

defined for all $r \geq 1$, $0 \leq k \leq n$.

As the definition of the shape functions takes a while to absorb, we describe the spaces in more elementary terms in the case of three dimensions.

- The space $\mathcal{S}_r\Lambda^0$, the polynomial shape functions for the H^1 space, consists of all polynomials u with *superlinear degree* $\operatorname{sdeg} u \leq r$. The superlinear degree of a monomial is its degree ignoring any variable that enters to the first power, and the superlinear degree of a polynomial is the maximum over its monomials. The criterion $\operatorname{sdeg} u \leq r$ was introduced in [2] to generalize the serendipity elements from 2 to n-dimensions.
- The space $\mathcal{S}_r\Lambda^1$, the shape functions for the $H(\text{curl})$ space, consists of vector fields of the form

$$(v^1, v^2, v^3) + \big(x^2x^3(w^2 - w^3), x^3x^1(w^3 - w^1), x^1x^2(w^1 - w^2)\big) + \operatorname{grad} u,$$

  with polynomials v^i, w^i, and u for which $\deg v^i \leq r$, $\deg w^i \leq r-1$, $\operatorname{sdeg} u \leq r+1$, and w^i is independent of the variable x^i.
- The $H(\text{div})$ space uses shape functions $\mathcal{S}_r\Lambda^2$, which are of the form

$$(v^1, v^2, v^3) + \operatorname{curl}\big(x^2x^3(w^2 - w^3), x^3x^1(w^3 - w^1), x^1x^2(w^1 - w^2)\big),$$

  with $\deg v^i \leq r$, $\deg w^i \leq r$, and w^i independent of the variable x^i.

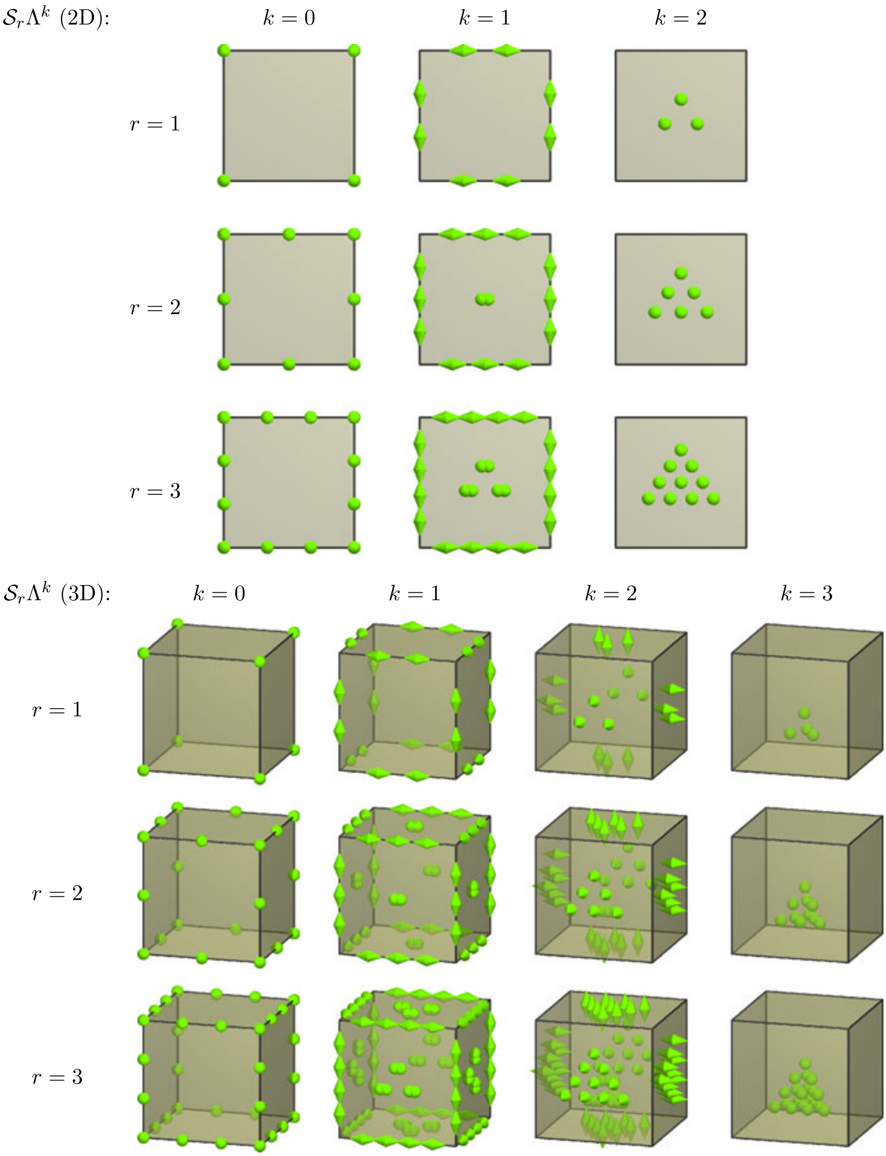


Fig. 5 The $\mathcal{S}_r\Lambda^k(\mathcal{T}_h)$ spaces in two and three dimensions

- Finally the L^2 space $\mathcal{S}_r\Lambda^3$ simply coincides with $\mathcal{P}_r$.

In [3] we establish the following properties of these spaces (in any dimension):

- degree property: $\mathcal{P}_r\Lambda^k(I^n) \subset \mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{P}_{r+n-k}\Lambda^k(I^n)$;
- inclusion property: $\mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{S}_{r+1}\Lambda^k(I^n)$;

Table 1 Dimension of $\mathcal{Q}_r^- \Lambda^k(I^n)$ and $\mathcal{S}_r \Lambda^k(I^n)$

k	r						r					
	1	2	3	4	5	6	1	2	3	4	5	6
$n=1$												
0	2	3	4	5	6	7	2	3	4	5	6	7
1	1	2	3	4	5	6	2	3	4	5	6	7
$n=2$												
0	4	9	16	25	36	49	4	8	12	17	23	30
1	4	12	24	40	60	84	8	14	22	32	44	58
2	1	4	9	16	25	36	3	6	10	15	21	28
$n=3$												
0	8	27	64	125	216	343	8	20	32	50	74	105
1	12	54	144	300	540	882	24	48	84	135	204	294
2	6	36	108	240	450	756	18	39	72	120	186	273
3	1	8	27	64	125	216	4	10	20	35	56	84
$n=4$												
0	16	81	256	625	1296	2401	16	48	80	136	216	328
1	32	216	768	2000	4320	8232	64	144	272	472	768	1188
2	24	216	864	2400	5400	10 584	72	168	336	606	1014	1602
3	8	96	432	1280	3000	6048	32	84	180	340	588	952
4	1	16	81	256	625	1296	5	15	35	70	126	210

- trace property: for each face f of I^n, $\operatorname{tr}_f \mathcal{S}_r \Lambda^k(I^n) \subset \mathcal{S}_r \Lambda^k(f)$;
- subcomplex property: $d\mathcal{S}_r \Lambda^k(I^n) \subset \mathcal{S}_{r-1} \Lambda^{k+1}(I^n)$.

The degrees of freedom for $\mathcal{S}_r \Lambda^k(T)$ are quite simple:

$$u \in \mathcal{S}_r \Lambda^k(T) \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)} \Lambda^{d-k}(f), \ f \in \Delta_d(T), \ d \geq k. \tag{4.1}$$

These are illustrated in Fig. 5. Notice that weighting function q is sought in a $\mathcal{P}_s$ space, not a $\mathcal{Q}_s$ space. Moreover, as the face dimension d increases by 1, the degree s of the space used for q decreases by 2. A major result of [3] is a proof that the degrees of freedom are unisolvent. Further, we show there that the resulting finite element spaces combine into de Rham subcomplexes with commuting projections:

$$\mathcal{S}_r \Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{S}_{r-1} \Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n} \Lambda^n(\mathcal{T}_h),$$

in which the degrees r decrease, as for the $\mathcal{P}_r \Lambda^K(\mathcal{T}_h)$ spaces on simplices.

For n-forms, the space $\mathcal{S}_r \Lambda^n(\mathcal{T}_h)$ is simply the discontinuous $\mathcal{P}_r$ space (but defined on boxes, rather than simplices). In 2-dimensions, the 0-form space $\mathcal{S}_r \Lambda^0(\mathcal{T}_h)$ is the well-known serendipity space, and the 1-form space is the rectangular BDM space defined in [10]. Hence these spaces were all known in 2 dimensions. However, in 3 and more dimensions they were not. The 0-form space is the appropriate generalization of the serendipity space to higher dimensions, a space first defined in 2011 [2]. The space $\mathcal{S}_r\Lambda^2$ in 3-D is, we believe, the correct analogue of the BDM elements to cubical meshes. It has the same degrees of freedom as the space in [9] but the shape functions have better symmetry properties. For 1-forms in 3-D, $\mathcal{S}_r\Lambda^1$ is a finite element discretization of $H(\text{curl})$. To the best of our knowledge, neither the degrees of freedom nor the shape functions for this space had been proposed previously. Finally, we note that the dimension of $\mathcal{S}_r^-\Lambda^k(T)$ tends to be much smaller than that of $\mathcal{Q}_r^-\Lambda^k(T)$, especially for r large, as can be observed in Table 1.

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