A NEW MIXED FORMULATION FOR THE NUMERICAL SOLUTION OF ELASTICITY PROBLEMS

Douglas N. Arnold
Department of Mathematics
University of Maryland
College Park, MD 20742
(301) 454-7066

Summary

A mixed formulation for boundary value problems in linear elasticity is presented. This formulation differs slightly from the classical Hellinger-Reissner formulation. The unknown fields are the displacement and a tensor related but not equal to the stress. The tensors appearing in the formulation need not be symmetric, and consequently mixed finite elements developed for scalar second order elliptic problems may be applied directly.

Introduction

This note reports on continuing work of the author and R. S. Falk of Rutgers University. A more complete account is in preparation.

The following notational conventions will be employed. Lower case letters with underlines are used to denote 3-vectors, lower case letters with double underlines denote 3\times3 tensors. Fourth order tensors are denoted by capital letters, their components by the corresponding lower case letters. The product of a fourth order tensor $\tau$ and a second order tensor is second order; thus $\tau = A\sigma$ means

$$\tau_{ij} = \sum_{k,l=1}^{3} a_{ijkl} \sigma_{kl}, \quad i,j = 1,2,3.$$  

If $X$ is a space of scalars, $\mathcal{X}$ denotes the space of vectors with components in $X$. If $\mathcal{Y}$ is a space of vectors, $\mathcal{Y}$ denotes the space of tensors with rows in $\mathcal{Y}$. The subspace of symmetric tensors in $\mathcal{Y}$ is denoted $\mathcal{Y}_s$. We will use the space $H(div,\mathcal{N})$ of square-integrable vector-valued functions on a domain $\mathcal{N}$ with square integrable divergences and the corresponding spaces $H(div,\mathcal{N})$ and $H(div,\mathcal{N}_s).

The system of (anisotropic, inhomogeneous) linear elasticity consists of the constitutive equations

$$\mathcal{G} = C\mathcal{E}(\mathcal{U})$$  

and the equilibrium equations

$$\text{div} \mathcal{G} = \mathcal{F}.$$  

These equations hold in the domain $\mathcal{N} \subset \mathbb{R}^3$ occupied by the elastic material and must be supplemented by appropriate boundary conditions. The vector-valued functions $\mathcal{U}$ and $\mathcal{F}$ give the displacement and imposed force, respectively, and the strain tensor $\mathcal{E}(\mathcal{U})$ is defined as $(\text{grad} \mathcal{U} + (\text{grad} \mathcal{U})^T)/2$. The coefficients of the elasticity tensor $C$ are given functions on $\mathcal{N}$ satisfying

$$c_{ijkl} = c_{klji} = c_{jkil}.$$  

Consequently the stress tensor $\mathcal{G}$ is symmetric. The elasticity tensor satisfies the positivity condition:

$$g_0 |\tau|^2 \leq \tau : \mathcal{C} \leq c_0 |\tau|^2 \quad \forall \tau \in \mathbb{R}^3,$$  

where $g_0$ and $c_0$ are positive constants (independent of the point $\mathcal{X} \in \mathcal{N}$ where the coefficients are evaluated) and

$$|\tau|^2 = \tau : \tau = \sum_{i,j} \tau_{ij}^2.$$  

Therefore, for each $\mathcal{X}$ the mapping $\tau \mapsto \mathcal{C} \tau$, viewed as a linear operator on the six dimensional space $\mathbb{R}^3$, is invertible. Its inverse may also be written as $\tau \mapsto A\tau$ with $A$ a fourth order tensor whose coefficients satisfy

$$a_{ijkl} = a_{klij} = a_{jikl},$$  

These coefficients form the compliance tensor. From (4) it follows that

$$y_1 |\tau|^2 \leq \tau : A\tau \leq y_2 |\tau|^2, \quad \forall \tau \in \mathbb{R}^3,$$  

with $y_1 = c_0^{-1}$, $y_2 = y_0^{-1}$ positive constants.

For simplicity, we consider the Dirichlet boundary condition $u = 0$ on $\partial \mathcal{N}$ but this restriction is inessential. To obtain a weak form of the resulting boundary value problem we invert the constitutive equations (1), multiply by a tensor $\tau \in H(div,\mathcal{N})$, and use the identity

$$\int_{\mathcal{N}} \mathcal{E}(\mathcal{U}) : \tau = -\int_{\mathcal{N}} \text{grad} \mathcal{U} : \tau = -\int_{\mathcal{N}} \mathcal{U} \cdot \text{div} \tau.$$  

The first equality holds since $\tau$ is symmetric, the second is Green's formula. Also testing the equilibrium equations against a function in $y \in L^2(\mathcal{N})$ we arrive at the weak formulation, which is known as the Hellinger-Reissner principle:

Find $(g, u) \in H^1(\mathcal{N}) \times L^2(\mathcal{N})$ such that

$$\int_{\mathcal{N}} A \tau : \mathcal{F} + \int_{\mathcal{N}} u \cdot \text{div} \tau = 0, \quad \forall \tau \in H(div,\mathcal{N}),$$  

$$\int_{\mathcal{N}} y \cdot \text{div} g = \int_{\mathcal{N}} f \cdot v, \quad v \in L^2(\mathcal{N}).$$  

(Note that we henceforth suppress explicit notation of the domain $\mathcal{N}$ from the function spaces and integrals.) The variational formulation (7,8) is mixed in that both stress and displacement fields are present.

To define a mixed finite element method based on this formulation we must specify finite element spaces $S_h \subset H(div,\mathcal{N}), V_h \subset L^2(\mathcal{N})$. The approximate solution $(g_h, u_h) \in S_h \times V_h$ is then determined by the equations analogous to (7,8) with $(\mathcal{F}, \mathcal{Y})$ restricted to $S_h \times V_h$. As is well-known the choice of the mixed finite elements
(i.e., of the spaces $S_h$ and $V_h$) is a delicate one:
the approximate solution need not approximate well even
if the finite element spaces afford good approximation.
Necessary and sufficient conditions for the quasiopti-
mal estimate
\[
\frac{\|u - u_h\|_L^2}{\|u - u_h\|_L^2} \leq c \inf_{v \in V} \frac{\|v - v_h\|_L^2}{\|v - v_h\|_L^2}
\]
are given by Brezzi [4]. In particular, in light of
(6) the following conditions are sufficient (the second is also necessary):
\[
\text{div } S_h \subset V_h^*, \quad (10)
\]
\[
\inf_{v \in V} \sup_{0 \neq v_h \in V_h} \frac{\|v \cdot \text{div } v_h\|_L^2}{\|v\|_L^2} \geq \gamma > 0. \quad (11)
\]
These conditions imply (9) with $c$ depending only on
$c_1$ and $\gamma_1$ in (6) and $\gamma$ in (11).

The analogous problem of finding convenient,
accurate, stable finite elements for mixed formulations of
scalar second order problems has been effectively
solved. Here one needs spaces $S_h \subset H(\text{div})$ (vector-
valued) and $V_h \subset L^2$ (scalar-valued) satisfying in
analogy to (10,11)
\[
\text{div } S_h \subset V_h^*, \quad (12)
\]
\[
\inf_{v \in V} \sup_{0 \neq v_h \in V_h} \frac{\|v \cdot \text{div } v_h\|_L^2}{\|v\|_L^2} \geq \gamma > 0. \quad (13)
\]
For example, we recall the definition of Raviart-Thomas-
Nedelec spaces of order $k \geq 0$, [11,12,15]. Let a
regular family of triangulations $\{h\}$ of $\Omega$ be
given, with meshsize $|h|$ tending to zero. Then the spaces
\[
S_h = \{v \in H(\text{div}) \mid V \in \Lambda_h, \exists p \in \mathbb{P}_k, q \in \mathbb{P}_k \}
\]
\[
\text{div } S_h = \{v \in L^2 \mid \forall \eta \in \Lambda_h, \exists p \in \mathbb{P}_k \}
\]
\[
V_h^* = \{v \in L^2 \mid \forall \eta \in \Lambda_h, \exists p \in \mathbb{P}_k \}
\]
satisfy (12,13) with $\gamma$ independent of $h$. These
spaces approximate $O(h^{k+1})$ in $H(\text{div}) \times L^2$,
and admit simple nodal bases. They have been success-
fully used in many computations, and thoroughly
analyzed [5-9,11-15]. For second order scalar elliptic
problems energy estimates, $L^2$ estimates in the
separate variables, negative norm estimates, $L^\infty$ estimates, superconvergence, and interior estimates
have all been shown.

Several authors have tried to adapt these elements to the elasticity equations, but such an
adaptation is not straightforward. The difficulty
arises from the requirement that $S_h$ consist of
symmetric tensors. (Equation (7) does not remain valid
for $\gamma$ asymmetric.) In particular one cannot simply
choose $S_h = \mathbb{S}_h \times S_h$ (the space of all tensors with
rows in the Raviart-Thomas-Nedelec space $S_h$) and
$V_h = V_h \times V_h$, although this choice clearly satisfies
(10,11).

A number of ways around this difficulty have been
proposed, at least in the plane homogeneous isotropic
case. Johnson and Mar Burger [10] developed a composite
linear triangular element sharing many, though not all,
of the desirable properties of the Raviart-Thomas ele-
ments and showed that it gave quasioptimal approxima-
family of composite elements of quadratic and higher
orders, with stability properties similar to those of the
Raviart-Thomas Nedelec elements and in particular
satisfying (10) and (11). Another approach was followed
by Arnold, Brezzi, and Douglas in [2]. In that paper
the symmetry of the stress tensor is enforced only
weakly via a Lagrange multiplier. Finite element spaces
for the stress, displacement, and Lagrange multiplier
based on, (although somewhat more complex than) the
lowest degree Raviart-Thomas spaces were proposed, and
the appropriate analogues of (10,11) were shown.

Each of these approaches apparently offers an
acceptable mixed finite element method for elasticity,
at least in the plane, but none fully shares the
simplicity and desirable convergence properties of the
Raviart-Thomas-Nedelec elements for scalar equations.
Here we propose a new mixed formulation for elasticity
in two or three dimensions to which the Raviart-Thomas-
Nedelec (or other) elements may be applied directly.
Our formulation differs slightly from (7,8). The
unknowns are the displacement $u$ and a tensor $\rho$ of
the form
\[
\rho = E \text{ grad } u^\gamma
\]
with the coefficient tensor $E$ derived from the elas-
ticity tensor as explained below. The tensor $\psi$ is
not symmetric and so does not coincide with the stress
tensor, but the stress components may be deduced from
the components of $\psi$ simply by linear combinations and
so this formulation preserves the advantageous property
of the usual mixed formulation that the stress may be
derived from the computed unknowns without differenti-
ing.

A New Mixed Formulation

We define the auxiliary variable $\rho$ by
\[
\rho = \sigma + \delta(\text{grad } u^\gamma) - \delta(\text{grad } u)\gamma^\top,
\]
where $\delta$ is the identity tensor. There is some free-
dom in the selection of the constant $\beta$; we take
$\beta = \gamma_0/3$ where $\gamma_0$ is a positive constant so that
(4) holds. Since
\[
\text{div}[\text{div } u^\gamma] = \text{grad } \text{div } u = \text{div } (\text{grad } u)^\top,
\]
the equilibrium equation (2) implies that $\text{div } \psi = f$.

Let us show how (14) may be inverted. Clearly
$\rho = (C + D) \text{ grad } u$ where $C$ is the elasticity tensor
and $D$ is defined by
\[
D = \text{tr}(r) \delta - r^\top.
\]
If $r$ is any tensor we may express it in terms of
three vectors, $x, y, z$ thus:

354
\[ \sigma = 2\nu c(u) + \lambda (\text{div} u) \delta, \quad \text{(16)} \]

where \( \mu > 0 \) and \( \lambda \geq 0 \) are the Lamé constants. In this case we choose \( \beta = \mu \) and (14) gives

\[ \rho = \mu \text{grad} u + (\nu + \lambda)(\text{div} u) \delta. \quad \text{(17)} \]

The analysis sketched in the last section shows that a mixed method based on our formulation will achieve quasioptimal approximation uniformly with respect to \( \mu \) and \( \lambda \) in any compact subinterval of \((0, \infty)\) and \((0, \infty)\) respectively. However as the material tends to incompressibility the coefficient \( \lambda \) tends to infinity, and one of the reasons that mixed methods are used for elasticity problems is that for many elements (any which satisfy (10) and (11) for example), the convergence is uniform with respect to \( \lambda \in (0, \infty) \). (In contrast the convergence of most displacement methods with low order elements degenerates in the case of a nearly incompressible material.) Let us recall how the uniformity with respect to \( \lambda \) of the standard mixed methods is proven and show that the same reasoning applies to our method.

First recall that despite the coefficient \( \lambda \) in (16), \( \sigma \) as well as \( u \), remains bounded and regular as \( \lambda \) tends to infinity. The quantity \( \lambda \text{ div } u \) tends to a limit, see, e.g., [3]. From (17) we see that \( \rho \) remains bounded and regular also.

For the compliance tensor in the isotropic case we have the equations

\[ \tau : A^\lambda = \frac{1}{2} |\tau|^2 - \frac{1}{2\nu(2u+3\lambda)} |\text{tr}(\tau)|^2. \quad \text{(18)} \]

Hence one may easily check that

\[ \tau : A^\lambda \geq \frac{1}{2\nu(2u+3\lambda)} |\tau|^2, \quad \tau \in \mathbb{R}^n. \]

The constant \( (2u+3\lambda)^{-1} \) appearing here is the best possible, and so the constant \( \gamma_1 \) in (6) tends to zero as \( \lambda \) tends to infinity. Nonetheless, as stated, the convergence of the mixed methods does not degenerate as \( \lambda \) tends to infinity. This is because dependence of the constant \( c \) in (9) on \( \gamma_1 \) can be weakened to dependence on \( \gamma_1 \) given by the bound

\[ \tau : A^\lambda \geq \gamma_1 |\tau|^2, \quad \tau \in \mathbb{R}^n. \quad \text{(19)} \]

where \( \gamma_1 \) is the deviatoric of \( \tau \). (For \( \tau \in \mathbb{H}(\text{div}) \) with \( \text{div} \tau = 0, \text{tr}(\tau) = 0 \), the \( L^2 \) norm of \( \tau \) is bounded by a constant multiple of the \( L^2 \) norm of \( \gamma_1 \). Hence (19) together with (10) is sufficient to establish the first Brezzi condition. See [3] for details.) Finally, since

\[ |\tau|^2 = |\tau|^2 + \frac{1}{3} |\text{tr}(\tau)|^2, \]

(18) implies (19) with \( \gamma_1 = 1/2\nu \) independent of \( \lambda \).

Now the tensor \( B \) inverting (17) is given by

\[ \text{grad } u = B \tau = \frac{1}{\nu} \tau - \frac{\mu}{\nu(4u+3\lambda)} \text{tr}(\tau) \delta. \]

Thus

\[ \text{grad } u = B \tau = \frac{1}{\nu} \tau - \frac{\mu}{\nu(4u+3\lambda)} \text{tr}(\tau) \delta. \]

\[ B \tau = \frac{1}{\nu} \tau - \frac{\mu}{\nu(4u+3\lambda)} \text{tr}(\tau) \delta. \]

\[ B \tau = \frac{1}{\nu} \tau - \frac{\mu}{\nu(4u+3\lambda)} \text{tr}(\tau) \delta. \]
\[
\tau : \mathbb{R}^2 - \frac{1}{\mu} \frac{\mu + \lambda}{\mu + 3\lambda} \frac{\text{tr}(\tau)}{2} \leq \frac{1}{\mu} \|
\tau \|_{\mathbb{R}^2}^2, \quad \tau \in \mathbb{R}^2
\]

The constant \( \mu \) in this bound does not depend on \( \lambda \). Consequently as long as the finite element spaces used with our formulation satisfy (10) the first condition of Brezzi holds uniformly in \( \lambda \). We conclude that the finite element approximation of an isotropic elastic material via the formulation (16,17) and spaces satisfying (10,11) is quasi-optimal uniformly with respect to the Lamé constant \( \lambda \).

An extension of the uniformity result to some classes of anisotropic materials will appear.

References


