The partial selective reduced integration method and applications to shell problems

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We shall briefly present here the main idea of partial selective reduced integration as developed in [1], [2]. As we shall see the idea is quite general and can be applied to a number of different situations. For the sake of simplicity, however, we shall concentrate on the particular case of Naghdi shell models. Once the idea is understood, its generalisation to other cases should be straightforward.

Let us consider the general expression of the energy to be minimised in the case of Naghdi shell models in the following way

$$J(U) = J_b(U) + \kappa\, J_m(U) + \kappa\, J_s(U) + J_e(U)$$

where the unknown U represents the three displacements u_1, u_2, u_3, and the two rotations θ_1, θ_2, and the four terms on the right hand side represent the bending, membrane, shear and the external portions of the energy, respectively. As it is well known, in the case of a very thin shell the coefficients of the membrane part and of the shear part are much bigger than the corresponding coefficients of the two other parts. This motivates, in the above expression, the introduction of the coefficient κ, that, after suitable scaling of the energy, we may assume to behave as t^{-2}, where t is the thickness of the shell.

It is clear that minimising $J(U)$ for a very small value of t (that is for a very big value of κ) is nearly equivalent to minimising $J_b(U) + J_e(U)$ under the constraints $J_m(U) = J_s(U) = 0$. In commonly used finite element approximations, these constraints are generally too restrictive: in fact the only finite element trial function that satisfies them is $U = 0$. Consequently, the resulting finite element solution is very close to zero even in the presence of strong external forces, unless the mesh size is small compared with the thickness (which is rarely the case if the thickness itself is very small). This phenonenon is commonly referred to as locking.

A common way to overcome locking is to use reduced integration in computing the integrals appearing in the expression of the energy $J(U)$. One then minimises

$$J_R(U) = J_{b,h}(U) + \kappa\, J_{m,h}(U) + \kappa\, J_{s,h}(U) + J_{e,h}(U)$$

where the subscript h indicates the use reduced integration. When the thickness is very small, we thus minimise $J_{b,h}(U) + J_{e,h}(U)$ under the constraint $J_{m,h}(U) = J_{s,h}(U) = 0$. If the reduced integration is properly done, this new constraint is far less restrictive than the previous one, and many finite element trial functions can pass it. However, this remedy brings its own difficulties. Of course, the use of an inexact integration scheme introduces inconsistency which can lead to slower convergence. More seriously the use of reduced integration may threaten the stability of the finite element method. This is because there may now exist nonzero finite element trial functions U^S for which $J_{m,h}(U^S) = J_{s,h}(U^S) = J_{b,h}(U^S) = 0$. Such functions are commonly referred to as spurious modes, and they were soon recognised to appear (quite often, unfortunately) when reduced integration was employed.

Selective reduced integration was introduced as a way to avoid spurious modes. Essentially selective reduced integration consists in using two different integration formulae: one for the shear and membrane part of the energy and a different one for the bending part of the energy (the integration formula for the external term is generally not crucial). Assuming, for the sake of simplicity, that the bending part is integrated exactly, we have in this case to minimise

$$J_{SR}(U) = J_b(U) + \kappa J_{m,h}(U) + \kappa J_{s,h}(U) + J_e(U).$$

In a number of similar situations, as for instance in elasticity problems with nearly incompressible materials, selective reduced integration proved to be very effective. Exact integration of the terms of the energy which don't involve the large parameter (the bending term in the case of shells) is sufficient to suppress the spurious modes, and so the use of reduced integration for the other terms is limited only by the need to avoid an excessive consistency error. In these situations, a choice of reduced integration (for some but not all terms of the energy) can be found which provides good schemes and accurate solutions. Selective reduced integration has been less successful in some other cases, however, for example for Reissner–Mindlin plates and for shell models. In these cases the exactly integrated terms of the energy are insufficient to control the spurious modes by themselves and the other terms are required. If the other terms are integrated too accurately, locking occurs. If they are not integrated accurately enough, spurious modes appear. It may be impossible to find a satisfactory scheme. In particular, although several reasonably good schemes for shells have been proposed in the literature (see for instance [3], [4] and [5]), the range of applicability of such schemes is somewhat uncertain, and no rigorous analysis is known to justify them.

This leads us to the idea of partial selective reduced integration. We recognise, on the one hand, that some form of reduced integration is required to treat the terms of the energy which contain large coefficients in order to avoid locking. On the other hand, we know that if these terms are completely underintegrated we may lose stability because of the appearance of spurious modes. The following compromise then suggests itself: one should split the relevant terms of the energy, like $\kappa\ J_m(U)$, as the sum of two terms

$$(\kappa\ -\ \alpha)\ J_{m,h}(U)\ +\ \alpha\ J_m(U),$$

(where α is a parameter to be chosen, and which will not be large) and to use reduced integration only for the first term of the splitting. The expression of the energy for partial selective reduced integration then reads

$$\begin{aligned} J_{PSR}(U) &= J_b(U) + (\kappa-\alpha)J_{m,h}(U) + \alpha\ J_m(U) \\ &\quad + (\kappa-\alpha)\ J_{s,h}(U) + \alpha\ J_s(U) + J_e(U)\,. \end{aligned}$$

Notice that the part of the energy with the (still) large coefficient $\kappa - \alpha$ is subintegrated in order to avoid locking, but that a part of the entire energy is integrated exactly in order to avoid spurious modes. It was shown in [1] for Reissner-Mindlin plate problems and in [2] for Naghdi shells that with a fixed choice of α (say $\alpha = 1$ to fix the ideas) one can choose an appropriate reduced integration rule which gives an accurate finite element solution without locking. The choice of the integration formula must still balance accuracy against locking, but the possible loss of stability associated with spurious modes is no longer a problem. This greatly simplifies the development of a good scheme. In particular for Naghdi shell problems under certain assumptions it was proven in [2] that the choice of seven nodes triangular elements (piecewise quadratics plus cubic bubbles) for both displacements and rotations, $\alpha = 1$, and, essentially, partial selective reduced integration with one point quadrature, provides a stable and first order accurate scheme. Much work remains to be done in order to have the most efficient schemes. For instance the results of [6] for plates suggest that the bubble function might be unnecessary, so a usual 6 nodes element may be sufficient. Similarly the results of [7] and [8] suggest that the choice of $\alpha = 1$ could be suboptimal and that the choice of a bigger value might be more convenient. In particular the numerical experiments of [7] and the theoretical analysis of [8] for plate problems suggest that the choice of $\alpha \simeq h^{-1}$ (h being the mesh size) is the most appropriate and gives rise to an error in the energy norm of the order $h^{3/2}$ instead of h. Again, much work is still to be done. We believe, however, that the basic idea of partial selective reduced integration can prove very fruitful for shells and for other similar problems.

References

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