A quick introduction to the Einstein equations

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The Einstein equations are simple geometrical equations to be satisfied by a metric of signature $- + + +$ on the 4-manifold representing spacetime. More specifically, they constrain the curvature tensor associated to the metric.

It is evident, geometrically, that there is a great deal of non-uniqueness in the Einstein equations.

If we coordinatize the manifold the equations can be viewed as 10 very complicated PDEs for the 10 component functions of the metric.
The coordinate-free viewpoint: geometry
Vector space concepts

$V$ an infinite dimensional vector space; $V^*$ its dual;

N.B.: there is a canonical identification $V \cong V^{**}$, but not $V \cong V^*$

tensor product $V \otimes W$;

$V \otimes W$ can be thought of as bilinear maps on $V^* \times W^*$ or linear maps from $V^*$ to $W$ or linear maps from $W^*$ to $V$

$$
\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l}
\cong \text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \rightarrow \mathbb{R}
$$

Since $V^* \otimes V$ is linear maps from $V \rightarrow V$,

$\exists \ \text{tr} : V^* \otimes V \rightarrow \mathbb{R}$ given by $\text{tr}(f \otimes v) = f(v)$
A (pseudo) inner product is a symmetric bilinear map $a : V \times V \to \mathbb{R}$ (so an element of $V^* \otimes V^*$) which is non-degenerate: $a(v, \cdot) \neq 0$ if $v \neq 0$.

Given an inner product we can assign every vector a squared length $a(v, v)$. It is not 0 if $v \neq 0$, but it can be negative.

Orthonormal basis: $a(e_i, e_j) = \pm \delta_{ij}$. The number of pluses and minuses is basis-independent, the signature of the inner product.

An inner product establishes a canonical identification $V \cong V^*$
Manifold concepts
Tensors on manifolds

$M$ an $n$-manifold, $p \in M$, $T_p M$ the tangent space of $M$ at $p$, $(T_p M)^*$ the cotangent space

\[ T_p^{(k,l)} M := T_p M \otimes \cdots \otimes T_p M \otimes T_p M^* \otimes \cdots \otimes T_p M^* \]

Maps $p \in M \mapsto v_p \in T_p^{(k,l)} M$, are called $(k, l)$-tensors

$(0, 0)$-tensors: functions $M \to \mathbb{R}$; $(1, 0)$-tensors: vector fields on $M$; $(0, 1)$-tensors: covector fields on $M$

A $(k, l)$-tensor is a machine that at each $p$ takes $k$ tangent covectors and $l$ tangent vectors and returns a number (multilinear in the (co)vectors, smooth in $p$).

All physical quantities in relativity are modeled as tensors.
Maps between manifolds

If \( \phi : M \to N \) is smooth and \( p \in M \), then
\[
d\phi_p : T_pM \to T_{\phi(p)}N
\]
is a linear map. For \( v \in T_pM \),
\[
d\phi_p v \in T_{\phi(p)}N
\]
is also denoted \( \phi_* v \), the push-forward of \( v \).

For \( I \) an interval about 0, \( \gamma : I \to M \) a curve, then
\[
\gamma'(0) := d\gamma_0 1
\]
is a tangent vector at \( \gamma(0) \).

If \( f : M \to \mathbb{R} \), then \( df_p \) is a linear map \( T_pM \) to \( \mathbb{R} \), i.e., \( df \) is a covector field.
A pseudo Riemannian metric is a symmetric, non-degenerate $(0, 2)$-tensor, i.e., at each point $p$, an inner product on $T_pM$.

The Einstein equations are concerned with assigning to a manifold a metric with signature $-\cdot\cdot$ properties.
Abstract index notation

For \((k, l)\)-tensors, use symbols adorned with \(k\) superscripts and \(l\) subscripts \(a, b, \ldots\).

\(v^a\) is a vector field, \(w_b\) is a covectorfield, \(R^d_{abc}\) is a \((1, 3)\)-field, etc.

The indices themselves have no meaning (like the \(\vec{v}\) in \(\vec{v}\)).

The tensor product of \(v^a_b\) and \(w^{ab}_c\) is written \(v^a_b w^{cd}_e\).
Counting sub- and superscripts shows it to be a \((3, 2)\)-tensor.

The trace of a \((1, 1)\)-tensor is indicated by a repeated index:
\(v^a_a\) (Repeated sub-/superscripts aren’t counted.)

\(v^{abc}_{ad}\) trace of a \((3, 2)\)-tensor wrt the first covector and vector variables, a \((2, 1)\)-tensor.
\[ v_{(ab)} := \frac{1}{2}(v_{ab} + v_{ba}), \text{ the symmetric part of } v_{ab} \]

\[ v_{[ab]} := \frac{1}{2}(v_{ab} - v_{ba}), \text{ the antisymmetric part of } v_{ab} \]

\[ v_{(ab)c} := \frac{1}{2}(v_{abc} + v_{bac}) \]

\[ v_{(abc)} := \frac{1}{6}(v_{abc} + v_{bca} + v_{cab} + v_{bac} + v_{cba} + v_{acb}) \]
Index lowering and raising

*If a metric* $g_{ab}$ *is specified*, we can identify a covector with a vector. We write $v_a$ for the vector identified with $v^b$:

$$v_a = g_{ab}v^b$$

This can apply to one index of many: $g_{ce}w_{ab}^{ed} = w_{abc}^d$,

or several: $g_{ce}g_{df}w_{ab}^{ef} = w_{abcd}$

Applied to the metric we find $g^b_a$ is the identity $\delta^b_a$, and $g^{ab}$ is the “inverse metric,” which can be used to raise indices:

$$v^a = g^{ab}v_b$$
Given a function $f : M \to \mathbb{R}$ and a vector $V^a \in T_pM$ there is a natural way to define the directional derivative $V^a \nabla_a f$:

$$V^a \nabla_a f(p) = \lim_{\epsilon \to 0} \frac{f(\"p + \epsilon V^a\") - f(p)}{\epsilon}.$$ 

By “$p + \epsilon V^a$” we mean $\gamma(\epsilon)$ where $\gamma : \mathbb{R} \to M$ is a curve with $\gamma(0) = p$, $\gamma'(0) = V^a$.

Thus $\nabla_a f$ is a covector field, which we previously called $df$.

It is not possible to define the directional derivative of a vector field $v^b$ in the same way, because $v^b(\"p + \epsilon V^a\") - v^b(p)$ involves the difference of vectors in different spaces.
If a metric $g_{ab}$ is specified, this determines a way to parallel transport a vector along a curve. Using this we can define $\nabla_a f^b$. Using the Leibnitz rule this easily extends to tensors of arbitrary variance. In this way we get a linear operator $\nabla$ from $(k, l)$-tensors to $(k, l + 1)$-tensors for all $k, l$. It satisfies the Leibniz rule, commutes with traces, gives the right result on scalar field, satisfies the symmetry

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f, \quad f : M \to \mathbb{R}$$

and is compatible with the metric:

$$\nabla_a g_{bc} = 0.$$ 

This characterizes the covariant differentiation operator.
It is not true that the second covariant derivative is symmetric when applied to vectors. Instead

\[(\nabla_a \nabla_b - \nabla_b \nabla_a)v^d = \frac{1}{2} R^d_{abc}v^c\]

for some tensor $R^d_{abc}$, called the Riemann curvature tensor.

$R_{(ab)cd} = 0$, $R_{abcd} = R_{cdab}$, $R_{[abc],d} = 0$

1 DOF in 2D, 6 in 3D, 20 in 4D

Bianchi identity: \[\nabla_{[a} R_{bc]de} = 0\]
The **Ricci tensor** is the trace of the Riemann tensor:

\[ R_{ab} = R^d_{adb} \]

The **scalar curvature** is its trace:

\[ R = R^a_a = g^{ab} R_{ab} \]

The **Einstein tensor** is

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \]

In 4D \( G_{ab} \) has the same trace-free part but opposite trace as \( R_{ab} \): Einstein is trace-reversed Ricci.

By the Bianchi identity,

\[ \nabla^a G_{ab} := g^{ac} \nabla_c G_{ab} = 0 \]
In a vacuum, the Einstein equations are simply

$$G_{ab} = 0$$

or $$R_{ab} = 0$$.

In GR we are interested in spacetimes, i.e., 4-manifolds endowed with a metric of signature $$- + + +$$ which satisfy the Einstein equations.

If matter is present, then $$G_{ab} = kT_{ab}$$ where the stress-energy tensor $$T_{ab}$$ comes from a matter model, $$k = \text{const.} = 8\pi G/c^4 = 2 \times 10^{-48} \text{ sec}^2/\text{g cm}$$
If $\phi : M \to N$ is any diffeomorphism of manifolds and we have a metric $g$ on $M$, then we can push forward to get a metric $\phi_*g$ on $N$. With this choice of metric $\phi$ is an isometry. It is obvious that the Riemann/Ricci/scalar/Einstein curvature tensors associated with $\phi_*g$ on $N$ are just the push-forwards of the those associated with $g$ on $M$. So if $g$ satisfies the vacuum Einstein equations, so does $\phi_*g$.

In particular we can map a manifold to itself diffeomorphically, leaving it unchanged in all but a small region. This shows that the Einstein equations plus boundary conditions can never determine a unique metric on a manifold.

Uniqueness can never be for more than an equivalence class of metrics under diffeomorphism.
The coordinate viewpoint: PDEs
Let \((x^1, \ldots, x^n) : M \to \mathbb{R}^n\) be a diffeomorphism of \(M\) (often only part of \(M\)) onto \(\Omega \subset \mathbb{R}^n\). At each point we can pull back the standard basis of \(\mathbb{R}^n\) to a basis for \(T_pM\). This coordinate-dependent choice of basis \((X^a_1(p), \ldots, X^a_n(p))\) at each point is the coordinate frame.

We also get a dual basis for \(T_pM^*\) and then a basis for all the \(T^{(k,l)}_pM\).

E.g., \(X^a_i(p) \otimes X^a_j(p), 1 \leq i, j \leq n\) gives a basis for \(T^{(2,0)}_pM\). We can expand a \((2,0)\)-tensor in this basis, and so represent it by an array of functions \(v^{ij} : \Omega \to \mathbb{R}\), called the components of the tensor.
Covariant differentiation in coordinates

If \( g_{ij} \) are the components of the metric and \( v^i \) are the components of some vector field \( v^b \), then the components of the covariant derivative \( \nabla_a v^b \) are

\[
\nabla_i v^j = \frac{\partial v^j}{\partial x^i} + \Gamma^j_{ik} v^k,
\]

where

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)
\]

are the Christoffel symbols of the metric in the particular coordinate system. Similar formulas exist for the covariant derivative of tensors of any variance.
Einstein equations in coordinates

\[(g^{ij}) = (g_{ij})^{-1}, \quad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \]

\[R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^m_{jk} \Gamma^l_{mi} - \Gamma^m_{ik} \Gamma^l_{mj} \]

\[R_{ij} = R_{iilj}, \quad R = g^{ij} R_{ij}, \quad G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \]

\[G_{ij} = kT_{ij} \]

10 quasilinear second order equations in 10 unknowns and 4 independent variables, 1000s of terms
Given a second coordinate system \((x'^1, \ldots, x'^n) : M \rightarrow \Omega'\) we get a second set of component functions \(g'_{ij}\) for the same metric.

\[
g_{ij}(x) = \frac{\partial \psi}{\partial x^i}(x) \frac{\partial \psi}{\partial x^j}(x) g'_{kl}(x'),
\]

where \(\psi\) is \(\Omega \rightarrow M \rightarrow \Omega'\).

\((g'_{ij})\) satisfies the vacuum Einstein equations iff \((g_{ij})\) does.

This suggests that roughly 4 of the 10 components \(g_{ij}\) can be specified independently of the Einstein equations.
Conclusions

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For computational (and other) purposes it is better to view the Einstein equations not as equations for a 4-metric but as equations for a 3-metric that evolves in time. Stay tuned. . .