Questions

(And some Answers)

(And lots of Opinions)

On Shell Theory

Shells

Q: What is a shell?

A: A three-dimensional elastic body occupying a thin neighborhood of a two-dimensional submanifold of $\mathbb{R}^3$. That is, a shell is a *physical object*. Our goal is to predict the displacement and stress (measurable physical quantities) arising in response to given loads and boundary conditions.

Q: Is a plate a shell?

A: Of course. Which is not to say that flat shells don't respond differently than, say, elliptical shells in certain regimes.

Shell models

Q: What is a shell model?

A: A systems of equations (usually PDEs) which, when solved, yields a displacement (and stress) field approximating that of the physical shell. So the "reconstruction" of the 3D field is an essential part of the model. Shell models can involve PDEs in 3 variables (like the equations of 3D elasticity) or be *dimensionally reduced* and involve only PDEs in 2 variables.

The mathematics of shells

Q: What can we mathematicians do with shell models?

- Derive them
- Analyze the behavior of the shell model
- Determine the accuracy of the shell model in relation to the shell
- Solve the shell model (numerically)

. . . just what this workshop is about.

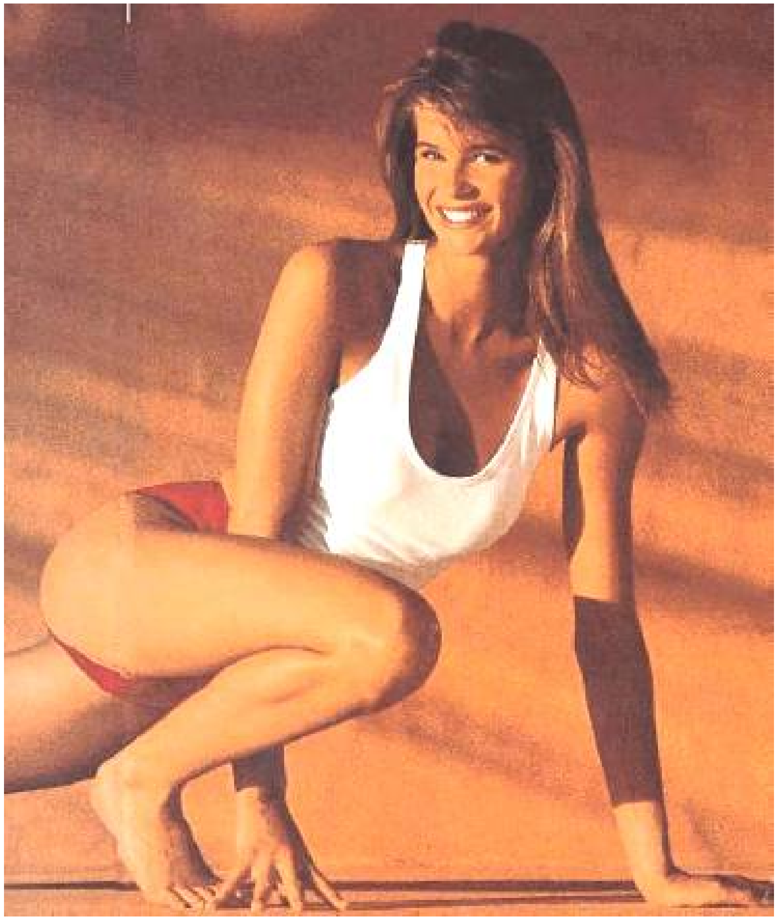
Q: How can mathematics, whose universe doesn't encompass actual physical objects, be used to derive a model for a physical shell or to prove that a shell model is accurate?

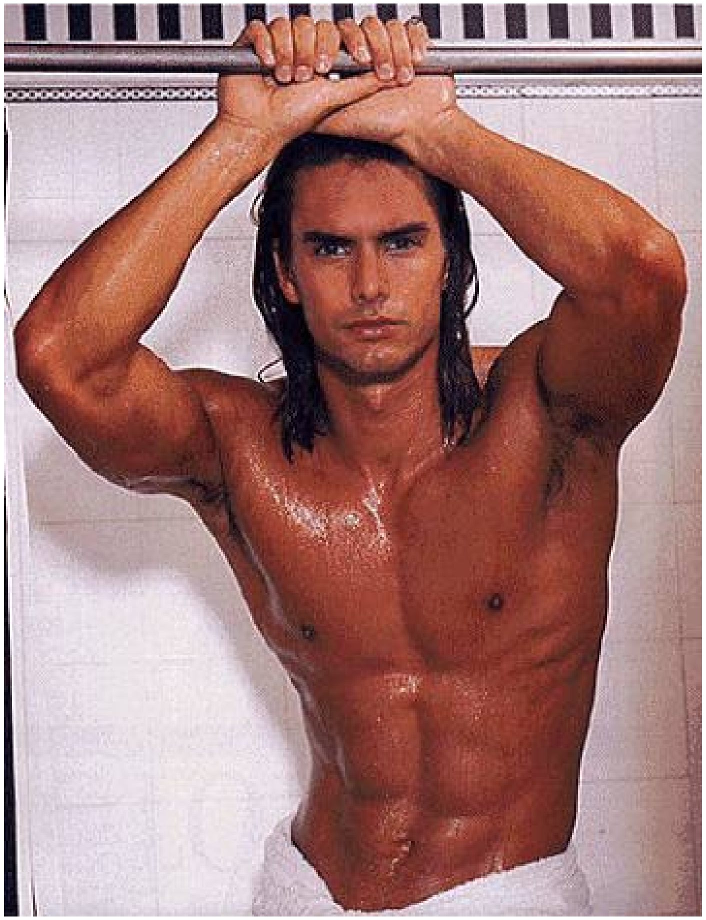
A: OK, OK, it can't. Relating the physical object to the mathematical model ultimately depends on non-mathematical reasoning: experiment, intuition, faith, . . .

But . . . mathematics can be very useful.

The trick is that you need a *supermodel*.

Supermodels





Supermodels

A supermodel is a mathematical model of a shell that we accept as sufficiently accurate. We then try to derive other shell models, in some sense simpler, which are close to the supermodel, and try to prove that the displacement field produced by the new model is close to that produced by the supermodel.

Typical supermodels are the system of 3D elasticity or linearized 3D elasticity.

Henceforth we will not distinguish between our supermodel (linear elasticity) and the true shell.

Derivation of shell models

The mathematical derivation of shell models from a supermodel is less important than obtaining bounds on the error.

While a systematic derivation is aesthetically satisfying and pedagogically sound, if a shell model is provably accurate, it doesn't really matter if it was derived via dubious principles.

Of course we may *need* mathematics to derive good models, but so far mechanical intuition is holding its own.

Derivation of shell models

All methods of derivations fall into one of three classes.

- Asymptotic methods Ciarlet, Destuynder, Lods, Miara, Raoult, Sanchez-Palencia, . . .
- Variational methods Arnold, Babuška, Falk, Madureira, Pitkäranta, Raoult, Schwab, Vogelius, . . .
- All other methods A cast of thousands . . .

The asymptotic approach

We start with the given shell of half-thickness ϵ_0, middle surface S^{ϵ_0}, elasticity tensor $\underline{\underline{\mathsf{C}}}^{\epsilon_0}$, surface loads $\underline{g}^{\epsilon_0}$, etc.

We then *invent* problems for each ϵ with half thickness ϵ, middle surface S^{ϵ}, elasticity tensor $\underline{\underline{\mathsf{C}}}^{\epsilon}$, surface loads $\underline{g}^{\epsilon}$, etc., coinciding with the given problem when $\epsilon = \epsilon_0$, and denote by $\underline{u}^{\epsilon}$ the displacement field solving the corresponding 3D problem.

E.g., $S^{\epsilon} = S^{\epsilon_0}$, $\underline{\underline{\mathsf{C}}}^{\epsilon} = \underline{\underline{\mathsf{C}}}$, $\underline{g}^{\epsilon} = (\epsilon/\epsilon_0)^p \underline{g}^{\epsilon_0}$.

We then determine, in some sense, $\lim_{\epsilon \downarrow 0} \underline{u}^{\epsilon}$. If we are lucky $\underline{u}^{\epsilon}$ can be determined from the solution of a 2D BVP.

There are infinitely many ways to embed the given shell into an asymptotic sequence, and so many possible limit models.

Q: How to choose one?

A: Basic rule of thumb: *all relevant nondimensionalized quantities in the given problem which are much greater or much smaller than* 1 *should be captured by the asymptotics.*

Example

If $$\frac{|\underline{g}^{\epsilon_0} \cdot \underline{n}|}{|\underline{g}^{\epsilon_0} \times \underline{n}|} = \frac{1}{100}$$ *don't* take $\underline{g}^{\epsilon} = \left(\frac{\epsilon}{\epsilon_0}\right)^p \underline{g}^{\epsilon_0}$.

That gives $|\underset{\sim}{g}^{\epsilon}|/|g_3^{\epsilon}| = O(1)$ as $\epsilon \to 0$, and misses a fundamental aspect of the given shell problem.

If $\epsilon_0 = 1/10$, better asymptotics uses

$$\underline{g}^{\epsilon} \cdot \underline{n} = \left(\frac{\epsilon}{\epsilon_0}\right)^2 \underline{g}^{\epsilon_0} \cdot \underline{n}, \quad \underline{g}^{\epsilon} \times \underline{n} = \underline{g}^{\epsilon_0} \times \underline{n}$$

A literary quiz

Q: Who said *"There are three kinds of lies: lies, damn lies, and statistics"*?

A: Mark Twain

Q: Why did he say it?

A: Because he wasn't familiar with asymptotic analysis.

The point: AA, like statistics, can be used to bring understanding to complex situations . . . or not.

Q: Is the partially clamped elliptic shell really a monster? Can more appropriate asymptotics tame the monster?

The variational approach

In the variational approach to deriving shell models,

1. We characterize the solution of the supermodel by a variational principle or weak formulation (minimum energy, minimum complementary energy, Hellinger–Reissner)

2. We then determine an approximation by restricting to a trial space of functions that are finite dimensional with respect to the transverse variable

Example: Reissner–Mindlin (a la AAFM '96)

1. Characterize $\underline{u}^\epsilon$, $\underline{\underline{\sigma}}^\epsilon$ as the unique critical point of the Hellinger–Reissner functional

$$(\underline{u}, \underline{\underline{\sigma}}) \mapsto \int_{\Omega^\epsilon} (\underline{\underline{\underline{\underline{\mathsf{A}}}}}\sigma : \sigma + \operatorname{div} \sigma \underline{u})$$

subject to the given traction boundary conditions on top and bottom.

2. Define $\underline{u}_R^\epsilon$, $\underline{\underline{\sigma}}_R^\epsilon$ as the critical point restricted to functions with the following polynomial degree in the transverse direction:

$$\deg \underline{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \deg \underline{\underline{\sigma}} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

The resulting model

$$\underline{\underline{u}}_R^\epsilon(\underset{\sim}{x}, x_3) = \begin{pmatrix} \underset{\sim}{\eta}^\epsilon(\underset{\sim}{x}) \\ \rho^\epsilon(\underset{\sim}{x})x_3 \end{pmatrix} + \begin{pmatrix} -x_3\underset{\sim}{\theta}^\epsilon(\underset{\sim}{x}) \\ w^\epsilon(\underset{\sim}{x}) + y^\epsilon(\underset{\sim}{x})r(x_3/\epsilon)) \end{pmatrix}$$

- $r(z) = (3/2)(z^2 - 1/5)$
- $\underset{\sim}{\eta}^\epsilon$ is determined by a plane elasticity problem
- ρ^ϵ is determined by $\underset{\sim}{\eta}^\epsilon$
- w^ϵ and $\underset{\sim}{\theta}^\epsilon$ are determined by the Reissner–Mindlin system
- y^ϵ is determined by w^ϵ and $\underset{\sim}{\theta}^\epsilon$

Stretching solution

Specifically

$$-2\epsilon\,\mathrm{div}\,\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}) = 2\underset{\sim}{\bar g} + \epsilon\frac{2\lambda}{2\mu+\lambda}\underset{\sim}{\nabla}(\tilde g_3 + \frac{\epsilon}{3}\mathrm{div}\,\underset{\sim}{\bar g}) \text{ in } \omega,$$

with

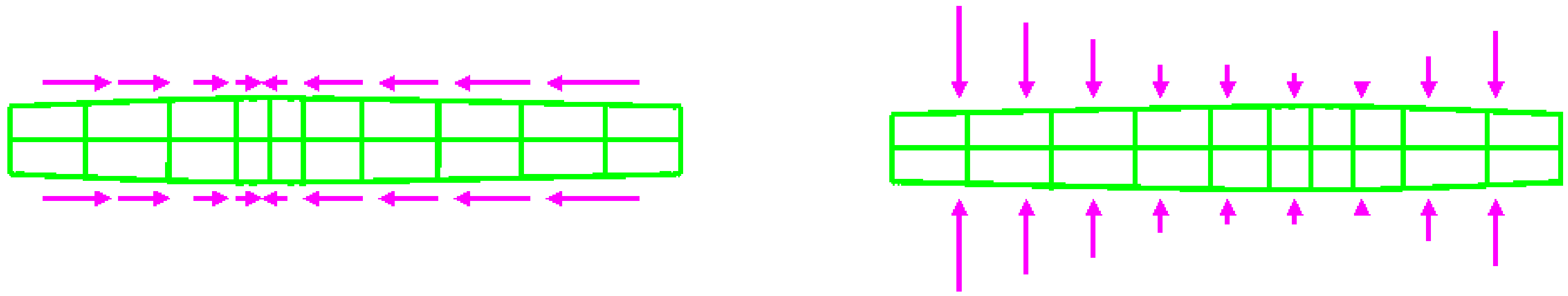
$$\tilde g_3(\underset{\sim}{x}) = \frac{1}{2}[g_3(\underset{\sim}{x},\epsilon) - g_3(\underset{\sim}{x},-\epsilon)], \quad \underset{\sim}{\bar g}(\underset{\sim}{x}) = \frac{1}{2}[\underset{\sim}{g}(\underset{\sim}{x},\epsilon) + \underset{\sim}{g}(\underset{\sim}{x},-\epsilon)]$$

Then $\rho^\epsilon = a_1\,\mathrm{div}\,\eta^\epsilon + a_2\tilde g_3 + a_3\epsilon\,\mathrm{div}\,\underset{\sim}{\bar g}$.

(a_i rational functions of μ and λ)

RM vs. KL for stretching

Note that the RM model is richer than the KL model. It takes into account that a plate can stretch in response to an odd transverse load and that a plate which is compressed will also expand in the transverse direction.



These effects, which are usually of higher order, are lost in the limit with the usual asymptotics.

Reissner–Mindlin bending solution

$$-\epsilon^3 \frac{2}{3} \operatorname{div} \underset{\approx}{\mathsf{C}}^* \underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) + \frac{5}{3}\epsilon\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) = \underset{\sim}{G}^\epsilon_R,$$

$$\frac{5}{3}\epsilon\mu \operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) = F^\epsilon_R,$$

where $\underset{\approx}{\mathsf{C}}^* \underset{\approx}{\tau} = 2\mu \underset{\approx}{\tau} + \lambda^* \operatorname{tr}(\underset{\approx}{\tau}) \underset{\approx}{\delta}$ and

$$\underset{\sim}{G}^\epsilon_R = -\frac{5}{3}\epsilon \underset{\sim}{\tilde{g}}^\epsilon - \frac{2}{15}\frac{\lambda}{2\mu+\lambda}\epsilon^2 \underset{\sim}{\nabla}(6\bar{g}^\epsilon_3 + \epsilon \operatorname{div} \underset{\sim}{\tilde{g}}^\epsilon),$$

$$F^\epsilon_R = \frac{\epsilon}{3} \operatorname{div} \underset{\sim}{\tilde{g}}^\epsilon + 2\bar{g}^\epsilon_3.$$

$$y^\epsilon = a_1\epsilon^2 \operatorname{div} \underset{\sim}{\theta}^\epsilon + a_2\epsilon\bar{g}^\epsilon_3 + a_3\epsilon^2 \operatorname{div} \underset{\sim}{\tilde{g}}^\epsilon$$

For *deriving* shell models, the variational approach is more natural than the asymptotic approach in several ways.

- It doesn't require that we embed the given shell problem in an infinite sequence of problems which don't really exist.
- It definitely reduces dimension: by its very nature its leads to a displacement field determined by finitely many functions of two variables.
- By construction it yields a displacement and/or stress field on the given physical plate, not on the midsurface or on an imaginary reference domain.
- If the resulting model is not accurate enough we can always improve it by using a larger trial space. *Hierarchical systems of models*.

For plates the asymptotic method leads to the fundamental Kirchhoff–Love model. But *only* this model. It doesn't seem to be possible to derive the better Reissner–Mindlin model, for instance. Taking more terms in the asymptotic expansion does not lead to a dimensionally reduced model.

The variational model leads to a precisely defined Reissner–Mindlin model (but *not* to KL), and also to a hierarchy of other models, including ones that had been derived before by other methods.

For shells, the asymptotic approach doesn't lead to any of the classical shell models, neither Koiter, Naghdi, Budiansky-Sanders, . . .

The variational approach for shells is not completely worked out. The work of Chapelle and Bathe fits here. Sheng Zhang derived a variant of Naghdi in this way.

Error estimates for shell models

Now let's go to error estimates, which are more important any way.

Error estimates for shell models: What?

The quantity to bound is the difference

$$\underline{u}_*^\epsilon - \underline{u}_M^\epsilon \quad \text{or} \quad \underline{\underline{\sigma}}_*^\epsilon - \underline{\underline{\sigma}}_M^\epsilon$$

A norm relevant to the given problem should be used. It will of course live on the physical domain. E.g.:

$$\|\underline{u}_*^\epsilon - \underline{u}_M^\epsilon\|_{\underline{L}^2(\Omega^\epsilon)}, \quad \|u_{*3}^\epsilon - u_{M3}^\epsilon\|_{L^2(\Omega^\epsilon)},$$

$$\int_{\Omega^\epsilon} \underline{\underline{\underline{\underline{\mathsf{C}}}}}\,\underline{\underline{e}}(\underline{u}_*^\epsilon - \underline{u}_M^\epsilon) : \underline{\underline{e}}(\underline{u}_*^\epsilon - \underline{u}_M^\epsilon)\, dx^\epsilon, \ldots$$

Since we don't know the size, we should use the *relative error*:

$$\frac{\|\underline{u}_*^\epsilon - \underline{u}_M^\epsilon\|_{\Omega^\epsilon}}{\|\underline{u}_*^\epsilon\|_{\Omega^\epsilon}}$$

An alternative is to post-process the 3D solution to obtain functions on the middle surface and compare these to the solution of the 2D PDEs. But this is less satisfying: it doesn't directly address the given problem of approximating the displacement and/or stress fields in the given shell.

"Ask not what the shell can do for your model, ask what your model can do for the shell."

Error estimates for shell models: How?

There are two main approaches:

- Asymptotic methods: embed the given shell problem into a family of problems with varying domains, varying loads, etc., parametrized by the shell thickness. Determine the first several terms of the asymptotic expansion, subtract the shell model solution, and compute the norms of the remaining terms.

- Variational methods: based upon the two energies principle.

The two energies principle

The 3D boundary value problem consists of

1. the constitutive equation $\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^{\epsilon}_{*}) = \underline{\underline{\sigma}}^{\epsilon}_{*}$
2. the equilibrium equation $\underline{\mathrm{div}}\,\underline{\underline{\sigma}}^{\epsilon}_{*} = \underline{f}^{\epsilon}$
3. the boundary conditions

If the model solution $\underline{u}^{\epsilon}_{M}$, $\underline{\underline{\sigma}}^{\epsilon}_{M}$ satisfies the equilibrium equation and boundary conditions exactly, the error can surely be bounded by the residual in the constitutive equation.

The two energies principle

In fact, if we use the energy norms, we have a simple identity

$$\|\underline{\underline{\mathsf{A}}}^{1/2}(\underline{\underline{\sigma}}_*^\epsilon - \underline{\underline{\sigma}}_M^\epsilon)\|^2_{L^2(\Omega^\epsilon)} + \|\underline{\underline{\mathsf{C}}}^{1/2}\underline{\underline{e}}(\underline{u}_*^\epsilon - \underline{u}_M^\epsilon)\|^2_{L^2(\Omega^\epsilon)} \\ = \|\underline{\underline{\sigma}}_M^\epsilon - \underline{\underline{\mathsf{A}}}\sigma_M^\epsilon\|^2_{L^2(\Omega^\epsilon)}$$

as long as $\underline{u}_M^\epsilon$ is *kinematically admissable* and $\underline{\underline{\sigma}}_M^\epsilon$ is *statically admissable*. (No ϵ-dependence.)

This is an *a posteriori* error estimate. It bounds the error in the model in terms of residual of the model solution, and so doesn't require detailed knowledge of the 3D solution.

We don't need to embed the problem in a parametrized family or to make assumptions on the load.

However, if we want to determine an order of convergence with respect to ϵ, than probably we will have to do so.

Q: Can Koiter's hope for a universal estimate be achieved?

Applicability of the variational approach

The two energies principle can be used to prove error estimates for any models, not just one derived by variational principles. In the 1950's Morgenstern used it to prove the first rigorous error convergence estimates for KL plate bending. He constructed admissable stress and displacement fields starting from a solution of the biharmonic equation and computed the constitutive residual. This approach leads to $O(\sqrt{\epsilon})$ convergence in relative energy norm.

The two energies principle is particularly suited to analyzing *complementary energy variational models*, which *automatically* yield statically admissable stress fields.

Applicability of the asymptotic approach

The asymptotic method can be used to analyze any models, not just the asymptotic limit ones. For other models, this requires that the model solution, as well as the 3D solution, be expanded in an asymptotic expansion. For example, in his thesis Madureira used the asymptotic method to estimate the errors in a hierarchical family of models for the Poisson equation on a thin plate. The estimates were in the form

$$\frac{\|\underline{u}_*^\epsilon - \underline{u}_p^\epsilon\|_{H^1(\Omega^\epsilon)}}{\|\underline{u}_*^\epsilon\|_{H^1(\Omega^\epsilon)}} \le \frac{C}{p^{-s}}\epsilon + O(\epsilon^2),$$

and so rigorously justify the value of higher order models.

Comparison and questions

The variational approach is relatively easy to carry out rigorously when it applies.

Q: But can it be used to get estimates in norms other than the energy norm?

The asymptotic approach requires detailed understanding of the the 3D solution, which can be hard to get and to make rigorous. But it gives more information and insight. Thanks to Dauge, Gruais, Rössle, . . . , the structure of the 3D solution on plates is in hand.

Q: How far is our understanding of the asymptotics of the 3D solution on shells (Dauge, Faou, Sanchez-P, . . .) from what is needed for good error estimates?

Example: RM vs. KL

Consider a surface loaded plate where

$$\underset{\sim}{g}^{\epsilon}(\underset{\sim}{x},\epsilon) = -\underset{\sim}{g}^{\epsilon}(\underset{\sim}{x},-\epsilon), \quad g_3^{\epsilon}(\underset{\sim}{x},\epsilon) = g_3^{\epsilon}(\underset{\sim}{x},-\epsilon)$$

(pure bending).

Suppose $\underset{\sim}{\tilde{g}}^{\epsilon} = \underset{\sim}{g}$, $\bar{g}_3^{\epsilon} = \epsilon g_3$ with $\underset{\sim}{g}$, g_3 H^1 functions independent of ϵ.

Let $\underline{u}_K^{\epsilon}$ be the Kirchhoff–Love solution and $\underline{u}_R^{\epsilon}$ the Reissner–Mindlin solution.

Example: RM vs. KL

Theorem. *For any nonzero $\bar{g} \in H^1(\omega)$ the Reissner–Mindlin solution satisfies*

$$\frac{\|\underset{\sim}{u}^\epsilon_* - \underset{\sim}{u}^\epsilon_R\|_{L^2(\Omega^\epsilon)}}{\|\underset{\sim}{u}^\epsilon_*\|_{L^2(\Omega^\epsilon)}} \le C\epsilon, \quad \frac{\|u^\epsilon_{*3} - u^\epsilon_{R3}\|_{L^2(\Omega)}}{\|u^\epsilon_{*3}\|_{L^2(\Omega^\epsilon)}} \le C\epsilon,$$

$$\frac{\|\underline{u}^\epsilon_* - \underline{u}^\epsilon_R\|_{E(\Omega^\epsilon)}}{\|\underline{u}^\epsilon_*\|_{E(\Omega^\epsilon)}} \le C\epsilon^{1/2}.$$

Theorem. *If $g_3 + \operatorname{div} \underset{\sim}{g} \neq 0$ then the same L^2 estimates hold for the Kirchhoff–Love solution and the energy estimate holds for the corrected Kirchhoff–Love solution. If however $g_3 + \operatorname{div} \underset{\sim}{g} = 0$, then the relative error in the Kirchhoff–Love solution doesn't go to zero no matter what norm is used.*

The proof consists of bounds on the error and lower bounds on the solution. The former can be obtained by asymptotic analysis or the two energies principle (the estimates for RM when $g_3 + \operatorname{div} \underset{\sim}{g} = 0$ are new).

To estimate the constitutive residual and to get lower bounds on the RM solution we use a sharp energy estimate (Arnold–Falk–Winter 1997)

Theorem. *For the mixed Reissner–Mindlin system*

$$-\operatorname{div} \underset{\approx}{\underset{\approx}{C}}\, \underset{\approx}{e}(\underset{\sim}{\theta}^{\epsilon}) + \underset{\sim}{\zeta}^{\epsilon} = \underset{\sim}{G}, \quad \operatorname{div} \underset{\sim}{\zeta}^{\epsilon} = F,$$

$$-\theta^{\epsilon} + \underset{\sim}{\nabla} w^{\epsilon} + \epsilon^2 \zeta^{\epsilon} = \underset{\sim}{J},$$

$$\|\underset{\sim}{\theta}^{\epsilon}\|_{H^1} + \|w^{\epsilon}\|_{H^1} + \|\zeta^{\epsilon}\|_{H^{-1}(\operatorname{div})\cap \epsilon L^2}$$
$$\approx \|\underset{\sim}{G}\|_{H^{-1}} + \|F\|_{H^{-1}} + \|\underset{\sim}{J}\|_{\mathring{H}(\operatorname{rot}) + \epsilon^{-1} L^2},$$

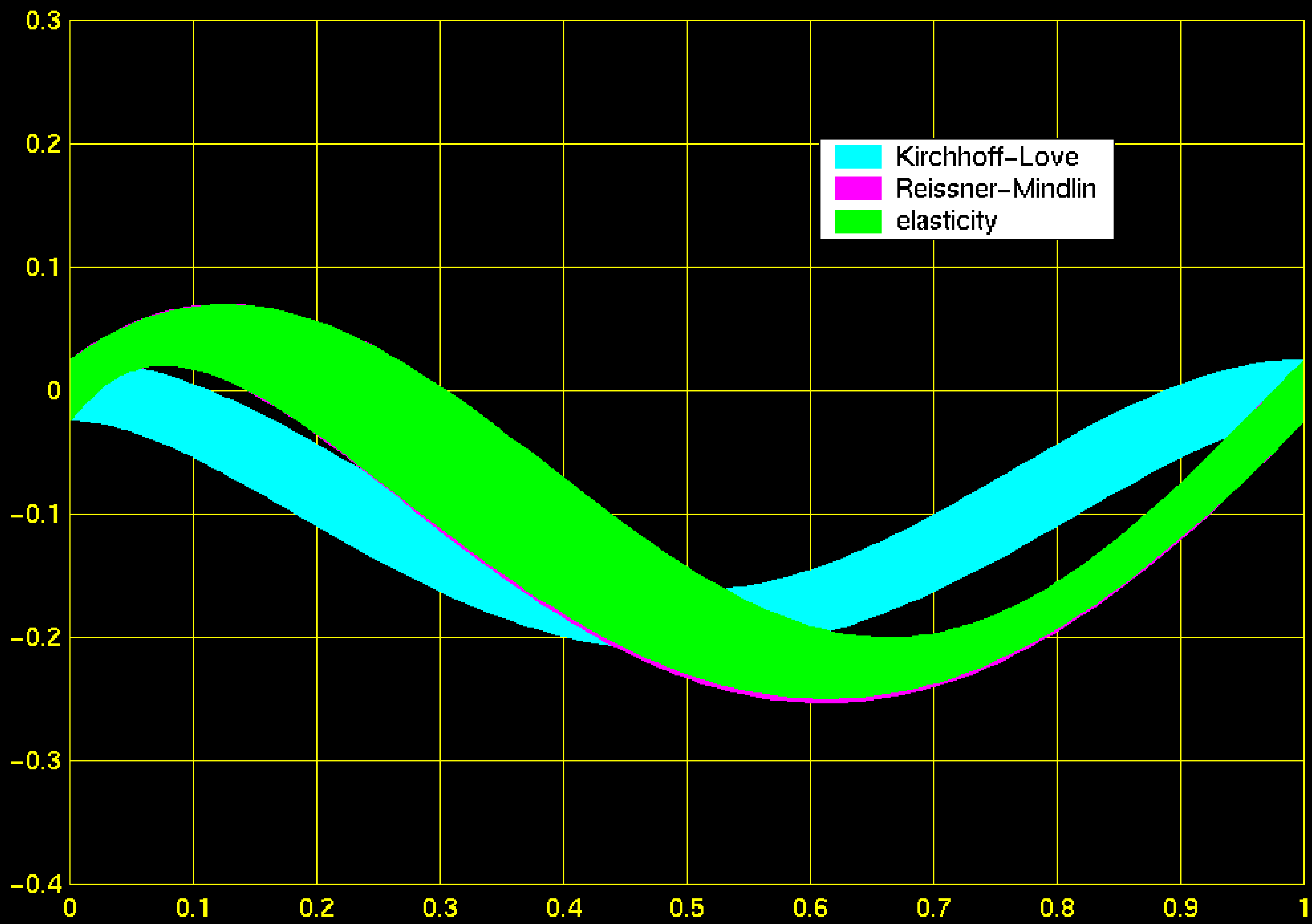
uniformly in ϵ.

Interpretation

When

$$\|\epsilon g_3^\epsilon + \operatorname{div} \underset{\sim}{g}^\epsilon\| \ll \max(\|\epsilon g_3^\epsilon\|, \|\underset{\sim}{g}^\epsilon\|),$$

the plate will undergo significant shear, and the Kirchhoff–Love solution, which ignores the shear, will not be accurate. While the convergence theorem applies, the asymptotic assumption on the loads is misleading.


Kirchhoff–Love
Reissner–Mindlin
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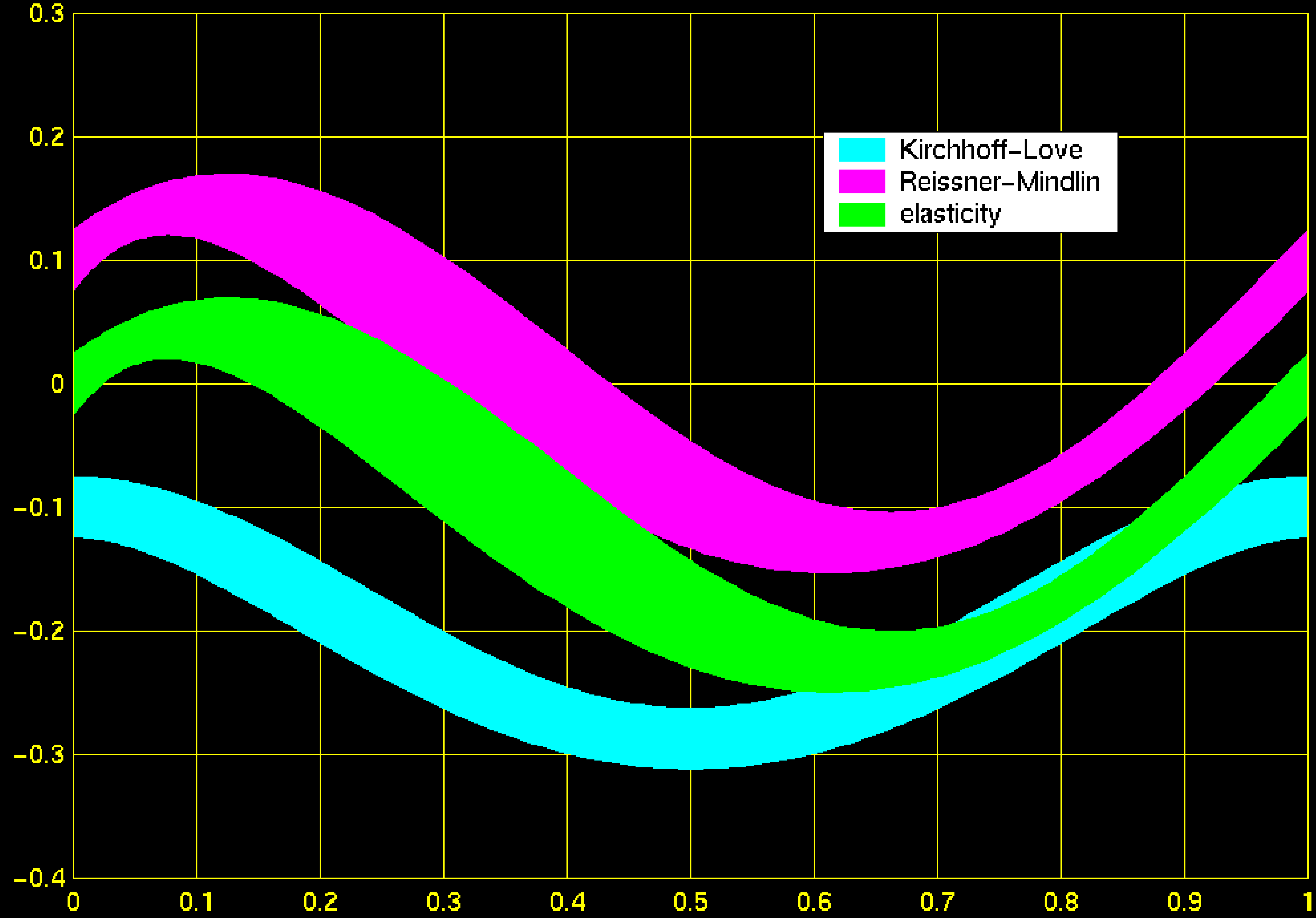

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This is one result addressing Philippe Ciarlet's challenge:

> *While it is generally agreed in computational mechanics circles that the Reissner–Mindlin theory is "better" than the Kirchhoff–Love theory, especially for "moderately thin plates," this assertion is not yet fully substantiated.*

It is surely not the whole answer. Reissner–Mindlin theory is also preferred because it better represents boundary conditions (it can distinguish between hard and soft simple support), and because it offers at least some approximation of the boundary layer. This is even more true for higher order hierarchical elements (Schwab–Wright, Madureira).

Similar things can be done with membrane loads for formally "bending dominated" (uninhibited) shells.

Questions, questions, questions . . .

Q: Can better asymptotics tame some of the monsters? Which 3D shells are really, physically, monsters?

Q: Can reasonable a priori estimates be proven without any a priori assumptions on the load?

Q: Can the variational approach be used to get estimates in norms other than the energy norm? Is there a duality argument? What are the best ϵ-independent estimates on the 3D problem?

Q: Asymptotic expansion of the 3D solution and error bounds?

Questions, questions, questions . . .

Q: Does the engineering approach of approximating $\underline{u}_*^\epsilon$ directly by $\underline{u}_{*h}^\epsilon$ have merit for analysis, or is it always better to define a dimensionally reduced model and use the triangle inequality?

Q: Are the goals of (1) consistent, stable approximation of the membrane energy in a shell model and (2) avoidance of locking when the membrane energy is constrained, really in contradiction? Does the p version provide a counterexample? Can a good h element be designed?

Questions, questions, questions . . .

Q: If you think a defintion of locking is hard to agree on, try "membrane-dominated"? 3D or 2D? Load dependent or not? Local or global? What really matters for computation?

Q: Does the Arnold–Brezzi estimate locking-free error estimate really require piecewise constant coefficients?

Q: How to interpret these locking-free estimates for membrane-dominated shells? Something is converging in a relative norm uniformly in ϵ: does it matter?

See you next week

. . . and at Elastic Shell Workshop 2010