1. Let \( v_1, \ldots, v_n \in \mathbb{R}^m \), all lengths \(|v_i|\) are not larger than 1. Let \( p_1, \ldots, p_n \in [0, 1] \) be arbitrary and set \( w = p_1 v_1 + \cdots + p_n v_n \). Then there exist \( \varepsilon_1, \ldots, \varepsilon_n \) each equal to 0 or 1, so that if \( v = \varepsilon_1 v_1 + \cdots + \varepsilon_n v_n \), then
\[
|w - v| \leq \frac{\sqrt{n}}{2}.
\]

Fix \( v_1, \ldots, v_n \in \mathbb{R}^m \) with \(|v_i| \leq 1\) and \( p_1, \ldots, p_n \in [0, 1] \), and set \( w = \sum_{i=1}^{n} p_i v_i \) as stated in the question.

For each \( i = 1, \ldots, n \), let \( X_i \) be the projection / indicator function
\[
X_i : \{0, 1\}^n \to \{0, 1\}
\]
\[
\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \mapsto \varepsilon_i.
\]

Next, let \( V = \sum_{i=1}^{n} X_i v_i \in \mathbb{R}^n \) and \( D = |w - V|^2 \in [0, \infty) \). Note that we haven’t put a probability space structure on \( \{0, 1\}^n \) yet. Once we do so, \( X_i \)’s and in turn \( V \) and \( D \) will become random variables. Since our aim is to show the existence of an \( \varepsilon \in \{0, 1\}^n \) with \( D(\varepsilon) \leq \frac{n}{4} \), it suffices to find a probability distribution on \( \{0, 1\}^n \) which satisfies \( \mathbb{E}(D) \leq \frac{n}{4} \).

Writing \( \langle -,- \rangle \) for the Euclidean inner product, we have
\[
D = \langle w - V, w - V \rangle = \left\langle \sum_{i=1}^{n} (p_i - X_i) v_i, \sum_{j=1}^{n} (p_j - X_j) v_j \right\rangle
\]
\[
= \sum_{i,j=1}^{n} (p_i - X_i)(p_j - X_j) \langle v_i, v_j \rangle
\]
\[
= \sum_{i=1}^{n} (p_i - X_i)^2 |v_i|^2 + \sum_{i \neq j} (p_i - X_i)(p_j - X_j) \langle v_i, v_j \rangle
\]
\[
\leq \sum_{i=1}^{n} (p_i - X_i)^2 + \sum_{i \neq j} (p_i - X_i)(p_j - X_j) \langle v_i, v_j \rangle.
\]

Now we impose the probability space structure on \( \{0, 1\}^n \) such that each \( \varepsilon_i \) is 1 with probability \( p \) and 0 with probability 1. Then \( X_i \) has the Bernoulli probability distribution \( B(p_i) \), so has expectation \( \mathbb{E}(X_i) = p_i \). Moreover, for \( i \neq j \) the random variables \( X_i \) and \( X_j \) are independent, so
\[
\mathbb{E} ((p_i - X_i)(p_j - X_j)) = \mathbb{E}(p_i - X_i)\mathbb{E}(p_j - X_j) = 0.
\]
Therefore, by the linearity of the expectation,
\[ \mathbb{E}(D) \leq \sum_{i=1}^{n} \mathbb{E}((p_i - X_i)^2) = \sum_{i=1}^{n} (p_i^2 - 2p_i \mathbb{E}(X_i) + \mathbb{E}(X_i^2)) \]
\[ = \sum_{i=1}^{n} (\mathbb{E}(X_i^2) - p_i^2) \]
\[ = \sum_{i=1}^{n} ((p_i \cdot 1^2 + (1 - p_i) \cdot 0^2) - p_i^2) \]
\[ = \sum_{i=1}^{n} (p_i - p_i^2). \]

Finally, by elementary calculus the global maximum of the function \( f(x) = x - x^2 \) on the interval \([0, 1]\) occurs at \( \frac{1}{2} \), hence \( f(x) \leq \frac{1}{4} \) for every \( x \in [0, 1] \). As a result,
\[ \mathbb{E}(D) \leq \sum_{i=1}^{n} f(p_i) \leq \sum_{i=1}^{n} \frac{1}{4} = \frac{n}{4}. \]

2. Suppose \( n \geq 2 \) and let \( H = (V, E) \) be an \( n \)-uniform hypergraph with \( |E| = 4^{n-1} \) edges. Show that there is a coloring of \( V \) by four colors so that no edge is monochromatic.

Let \( C \) to be a set of four colors. Put the uniform probability distribution on \( C \), and the product distribution on \( C^V \). Since the set \( C^V \) can be identified with the set of colorings of the vertices with four colors (functions from \( V \) to \( C \)), informally this means that given \( c \in C \), a vertex has a \( \frac{1}{4} \)-chance to be colored with \( c \).

Note that every element of \( E \) is a subset of \( V \) with \( |e| = n \). In particular, the vertex coloring \( f \in C^V \) making \( e \) monochromatic means that \( f(e) \) is a singleton. For every \( e \in E \), consider the random variable
\[
X_e : C^V \to \{0, 1\}
\]
\[ f \mapsto \begin{cases} 1 & \text{if } f(e) \text{ is a singleton}, \\ 0 & \text{otherwise}. \end{cases} \]

So \( X_e = 1 \) if \( e \) is monochromatic and \( 0 \) otherwise. Setting \( X = \sum_{e \in E} X_e \), for \( f \in C^V \), \( X(f) \) is the number of monochromatic edges induced by the coloring \( f \). We want to show that there exists \( f \) such that \( X(f) = 0 \). First,
\[ \mathbb{E}(X) = \mathbb{E}\left( \sum_{e \in E} X_e \right) = \sum_{e \in E} \mathbb{E}(X_e) \]
\[ = \sum_{e \in E} \mathbb{P}(f(e) \text{ is a singleton}) \]
\[ = \sum_{e \in E} \sum_{c \in C} \mathbb{P}(f(e) = \{c\}) \]
\[ = \sum_{e \in E} \sum_{c \in C} \left( \frac{1}{4} \right)^n = |E||C|4^{-n} \]
\[ = 4^{n-1} \cdot 4 \cdot 4^{-n} = 1. \]
Second, choosing \( f_0 \in C^V \) to be a constant function (every vertex is colored the same), we have \( X(f_0) = |E| = 4^{n-1} > 1 \) as \( n \geq 2 \). Since \( \mathbb{E}(X) = 1 \), we deduce that there exists an \( f \in C^V \) such that \( X(f) < 1 \), that is, \( X(f) = 0 \).

3. Let \( F \) be a family of subsets of \( N = \{1, \ldots, n\} \) and suppose there are no \( A, B \in F \) satisfying \( A \subset B \). Let \( \sigma \in S_n \) be a random permutation of elements of \( N \) and consider the random variable \( X \) which is the number of \( i \)-s such that \( \{\sigma(1), \ldots, \sigma(i)\} \in F \). By considering the expectation of \( X \) prove that

\[
|F| \leq \binom{n}{\lfloor n/2 \rfloor}
\]

(here \( \lfloor x \rfloor \) denotes the integer part of \( x \)).

Let \( X_i \) be the indicator function defined by

\[
X_i(\sigma) = \begin{cases} 
1 & \text{if } \{\sigma(1), \ldots, \sigma(i)\} \in F, \\
0 & \text{otherwise}.
\end{cases}
\]

Observe that by the condition on \( F \), at most one of \( X_1(\sigma), \ldots, X_n(\sigma) \) is an element of \( F \) for each \( \sigma \). Thus \( X = \sum_{i=1}^n X_i \) and in particular \( X \leq 1 \).

Thus, no matter what probability space structure we impose on the set of permutations \( S_n \), using the additivity of the expectation we get

\[
1 \geq \mathbb{E}(X) = \mathbb{E} \left( \sum_{i=1}^n X_i \right) = \sum_{i=1}^n \mathbb{E}(X_i).
\]

Now we choose uniform distribution on \( S_n \), that is, every permutation is equally probable. Then for each \( i \), writing \( N_i = \{A \in N : |A| = i\} \) and \( F_i = F \cap N_i \) (so \( F_i \) consists of \( i \)-element subsets of \( N \) contained in \( F \)), we have

\[
\mathbb{E}(X_i) = \mathbb{P}(\{\sigma(1), \ldots, \sigma(n)\} \in F) = \frac{|F_i|}{|N_i|} = \frac{|F_i|}{\binom{n}{i}}.
\]

By using the fact \( \binom{n}{i} \leq \binom{n}{\lfloor n/2 \rfloor} \) for any \( i = 1, \ldots, n \), we conclude that \( \mathbb{E}(X_i) \geq \frac{|F_i|}{\binom{n}{\lfloor n/2 \rfloor}} \).

Hence

\[
1 \geq \mathbb{E}(X) \geq \sum_{i=1}^n \frac{|F_i|}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|F|}{\binom{n}{\lfloor n/2 \rfloor}},
\]

so \( |F| \leq \binom{n}{\lfloor n/2 \rfloor} \).

4. Let \( G = (V,E) \) be a bipartite graph with \( n \) vertices and a list \( S(v) \) of more than \( \log_2 n \) for each vertex \( v \in V \). Prove that there is a proper coloring of \( G \) assigning to each vertex \( v \in V \) a color from its list \( S(v) \).

Let \( S = \bigcup_{v \in V} S(v) \). Put the uniform probability distribution on the two-element set \( \{C, D\} \), and then the product distribution on \( \{C, D\}^S \).
Since $G$ is bipartite, there exists $A \subseteq V$ such that every edge in $G$ is in between $A$ and $B := V - A$. For each $a \in A$, define a random variable 
\[
Y_a : \{C, D\}^S \rightarrow \{0, 1\}
\]
\[
f \mapsto \begin{cases} 
1 & \text{if } f(S(a)) = \{D\}; \\
0 & \text{otherwise.}
\end{cases}
\]
Similarly, for each $b \in B$, define a random variable 
\[
Z_b : \{C, D\}^S \rightarrow \{0, 1\}
\]
\[
f \mapsto \begin{cases} 
1 & \text{if } f(S(b)) = \{C\}; \\
0 & \text{otherwise.}
\end{cases}
\]
Note that for every $a \in A$, 
\[
\mathbb{E}(Y_a) = \mathbb{P}(f(S(a)) = \{D\}) = \left(\frac{1}{2}\right)^{|S(a)|} < \left(\frac{1}{2}\right)^{\log_2 n} = \frac{1}{n}.
\]
Similarly, $\mathbb{E}(Z_b) < \frac{1}{n}$ for each $b \in B$. Thus letting $X = \sum_{a \in A} Y_a + \sum_{b \in B} Z_b$, we have 
\[
\mathbb{E}(X) = \sum_{a \in A} \mathbb{E}(Y_a) + \sum_{b \in B} \mathbb{E}(Z_b) < |A| \frac{1}{n} + |B| \frac{1}{n} = \frac{|V|}{n} = 1.
\]
Therefore, there exists $f \in \{C, D\}^S$ such that $X(f) = 0$. This means that $Y_a(f) = 0$ for every $a \in A$ and $Z_b(f) = 0$ for every $b \in B$. In turn, this means $C \in f(S(a))$ and $D \in f(S(b))$ for every $a \in A$ and $b \in B$, respectively. Thus, for every $v \in V$, there exists a color $c_v \in S(v)$ such that 
\[
f(c_v) = \begin{cases} 
C & \text{if } v \in A, \\
D & \text{if } v \in B.
\end{cases}
\]
In particular, for every $a \in A$ and $b \in B$, $c_a \neq c_b$. Since every edge in $G$ is between $A$ and $B$, we conclude that the choice of $c_v$’s is a a proper coloring.

7. Let $F$ be a finite collection of sequences of 0-s and 1-s of finite length. Assume no member of $F$ is a beginning of a different member of $F$. For example, $F$ cannot contain both 101 and 10. Let $N_i$ denote the number of sequences of length $i$ in $F$. Prove that 
\[
\sum_{i} \frac{N_i}{2^i} \leq 1.
\]
Let $\ell$ be the length of the longest sequence in $F$. So if we let $Q_i$ to be the set of all sequences of length $i$, we have the decomposition 
\[
F = \bigcup_{i=1}^{\ell} (F \cap Q_i)
\]
where $N_i = |F \cap Q_i|$. By our choice of $\ell$, we have $N_k = 0$ when $k > \ell$.
Given a sequence $q$ of 0-s and 1-s, we write $T_i(q)$ for the $i$-th truncation of $q$. So for instance $T_3(00101) = 001$. 
Now, put the uniform distribution on the set $Q_\ell$. For each $i \in \{1, \ldots, \ell\}$ consider the random variable

$$X_i : Q_\ell \to \{0, 1\}$$

$$q \mapsto \begin{cases} 1 & T_i(q) \in F, \\ 0 & T_i(q) \notin F. \end{cases}$$

Given $q \in Q_\ell$, by the assumption on $F$, at most one of $X_1(q), \ldots, X_\ell(q)$ can be equal to 1. Thus defining $X = \sum_{i=1}^\ell X_i$, we have

$$1 \geq \mathbb{E}(X) = \sum_{i=1}^\ell \mathbb{E}(X_i) = \sum_{i=1}^\ell \frac{|F \cap Q_i|}{|Q_i|} = \sum_{i=1}^\ell \frac{N_i}{2^i} = \sum_{i=1}^\infty \frac{N_i}{2^i}. $$