1.2.10. Let $A$ be an integral domain, and $B$ its integral closure in the field of fractions $\text{Frac}(A)$. Suppose that $B$ is a finitely generated $A$-module. Show that $B$ is flat over $A$ if and only if $B = A$. One can show that this result is true without the assumption of finiteness of $B$ over $A$.

First we prove a lemma.

**Lemma 1.** Let $C$ be any subring of $\text{Frac}(A)$ containing $A$. Then $C$ is free as an $A$-module if and only if $C = A$.

**Proof.** The “if” part is trivial. So assume $C$ is free as an $A$-module. Suppose rank$_A C > 1$; then there exists at least two elements $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in C$ which are linearly independent over $A$. Then since

$$a_2b_1 \cdot \frac{a_1}{b_1} - a_1b_2 \cdot \frac{a_2}{b_2} = 0,$$

either $a_2b_1 = 0$ or $a_1b_2 = 0$. But $A$ is a domain and $b_1, b_2 \neq 0$ so either $a_1 = 0$ or $a_2 = 0$, contradicting linear independence. Thus rank$_A C = 1$. So there exists $c \in C \setminus \{0\}$ which generates $C$ as an $A$-module. So in particular $c^2 = ac$ for some $a \in A$. Cancelling $c$’s yields $c = a$ thus $C = A$. \hfill \square

We prove the claim of the question first assuming $A$ is local. In this case, being finitely generated and flat over $A$, as an $A$-module $B$ must be free hence $B = A$ by Lemma 1.

For the general case, we will show that the inclusion $\iota : A \hookrightarrow B$ is an $A$-module isomorphism. For this it suffices to show that for every $\mathfrak{p} \in \text{Spec} A$ the $A_{\mathfrak{p}}$-module homomorphism $\iota_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism.

Note that since localization is exact, the local domain $A_{\mathfrak{p}}$ can be identified as a subring of the algebra $B_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ can be identified as a subring of $(\text{Frac}(A))_{\mathfrak{p}} = \text{Frac}(A) = \text{Frac}(A_{\mathfrak{p}})$ (the equalities hold because $\text{Frac}(A)$ is nothing but the localization of $A$ at the prime ideal $0$ which is contained in $\mathfrak{p}$). And since $\iota$ is a finite homomorphism, so is $\iota_{\mathfrak{p}}$. And by Proposition 2.13 part (ii), $\iota_{\mathfrak{p}}$ is a flat homomorphism. Thus by the local case, we get that $\iota_{\mathfrak{p}}$ is an isomorphism.

1.3.12. Let $A$ be a Noetherian ring, $I$ an ideal of $A$, and $\hat{A}$ the $I$-adic completion of $A$. Let $a \in A$. Show that if $a$ is not a zero divisor in $A$, then it is not a zero divisor in $\hat{A}$. However, show by giving an example, that $A$ can be an integral domain without $\hat{A}$ being one.

Recall that the $I$-adic completion on modules defines a functor

$$\hat{\cdot} : A\text{-Mod} \to \hat{A}\text{-Mod}$$
such that given an $A$-module homomorphism $f : M \to N$, $\hat{f} : \hat{M} \to \hat{N}$ is the unique $\hat{A}$-module homomorphism which makes the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
\hat{M} & \xrightarrow{\hat{f}} & \hat{N}
\end{array}
$$

commute, where the vertical maps are the completion maps. Note that since $A$ is a commutative ring, every $a \in A$ defines an $A$-module homomorphism $m_a : A \to A$

$$
b \mapsto ab.
$$

(Additivity is clear and for scalar multiplication check $m_a(bc) = abc = bac = bm_a(c)$)

Writing $\hat{a}$ for the image of $a$ in $\hat{A}$ similarly we have a “multiply by $\hat{a}$” map $m_{\hat{a}} : \hat{A} \to \hat{A}$ which is a $\hat{A}$-module homomorphism. Since

$$
m_{\hat{a}}(\hat{b}) = \hat{a}\hat{b} = \hat{ab} = \hat{m_a}(b)
$$

by the above universal property we conclude that $\hat{m}_a = m_{\hat{a}}$.

Observe that the question asks us to show the implication “$m_a$ is injective” $\Rightarrow$ “$m_{\hat{a}}$ is injective”. Now by Corollary 3.14, the completion functor (°) is naturally isomorphic to $- \otimes \hat{A} : A\text{-Mod} \to \hat{A}\text{-Mod}$ on finitely generated $A$-modules. By Theorem 3.15 $\hat{A}$ is flat over $A$; hence the completion functor is exact on finitely generated modules. Thus if $m_a$ is injective, so is $\hat{m}_a = m_{\hat{a}}$.

**Proposition 2.** Let $I, J$ be coprime ideals in $A$, that is, $I + J = A$. Then the $IJ$-adic completion $A_{IJ}$ is isomorphic to $\hat{A}_I \times \hat{A}_J$.

**Proof.** By the Chinese remainder theorem, the natural map

$$
A/IJ \to A/I \times A/J
$$

is a ring isomorphism. Since $I^n$ and $J^n$ are also coprime, we have an isomorphism

$$
A/(IJ)^n = A/I^n \times A/J^n
$$

These isomorphisms are clearly compatible with the projections $A/(IJ)^{n+1} \to A/(IJ)^n$, $A/I^{n+1} \to A/I^n$, $A/J^{n+1} \to A/J^n$. Thus we obtain an isomorphism between the inverse limits

$$
\hat{A}_{IJ} = \lim_{\leftarrow} A/(IJ)^n \cong \lim_{\leftarrow} (A/I^n \times A/J^n) \cong \lim_{\leftarrow} A/I^n \times \lim_{\leftarrow} A/J^n = \hat{A}_I \times \hat{A}_J
$$

Therefore if we can find a domain $A$ who has coprime ideals $I, J$ such that $\hat{A}_I \neq 0$ and $\hat{A}_J \neq 0$ then the $IJ$-adic completion of $A$ won’t be a domain; exhibiting an example as the question wants. Indeed we can take $A = \mathbb{Z}$, $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$ (then $\hat{A}_I = \mathbb{Z}_2$ and $\hat{A}_J = \mathbb{Z}_3$).

**2.1.4.** Let $A$ be a ring.

(a) Let $p$ be a minimal prime ideal of $A$. Show that $pA_p$ is the nilradical of $A_p$. Deduce from this that every element of $p$ is a zero divisor in $A$. 
We know that in general for a multiplicative subset $S$ of $A$ the prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of $A$ that do not intersect $S$. When $S = A - p$ this means that the prime ideals of $A_p$ are in one-to-one correspondence with the prime ideals of $A$ contained in $p$. Since $p$ is minimal, this means that $pA_p$ is the only prime ideal of $A_p$, hence equal to the nilradical of $A_p$.

Given $a \in p$, the above yields that $a^n/1 = 0 \in A_p$ for some $n \in \mathbb{N}$ so there exists $s \in A - p$ such that $sa^n = 0$ in $A$. Hence the set

$$\{n \in \mathbb{N} : sa^n = 0 \text{ for some } s \in S\}$$

is nonempty, therefore has a smallest element $r \in \mathbb{N}$. So $ta^r = 0$ for some $t \in S$ and by the minimality of $r$ we have $ta^{r-1} \neq 0$. But $(ta^{r-1})a = 0$ so $a$ is a nilpotent.

**(b)** Show that if $A$ is reduced, then any zero divisor in $A$ is an element of a minimal prime ideal. Show with an example that this is false if $A$ is not reduced (use Lemma 1.6). See also Corollary 7.1.3(a).

Let $a \in A$ be a zero divisor. So there exists $0 \neq b \in A$ such that $ab = 0$. Since $A$ is reduced, the multiplicative subset $S = \{1, b, b^2, \cdots\}$ does not contain 0 and hence $S^{-1}A$ is nonzero. Note that $a$ maps to 0 under the ring homomorphism $A \to S^{-1}A$ because

$$\frac{b}{1} \cdot \frac{a}{1} = 0$$

and $\frac{b}{1}$ is a unit in $S^{-1}A$. Now let $q$ be a minimal prime ideal of $S^{-1}A$ (nonzero rings always have minimal primes by Zorn’s lemma). Then $q = S^{-1}p$ for some prime ideal $p$ of $A$ which does not intersect $S$ and is minimal among prime ideals with this property. Clearly it follows that $p$ is (unconditionally) a minimal prime ideal of $A$. And since $0 \in q$, $a$ must lie in $p$.

For a counterexample let $k$ be any field and let $A = k[X,Y]/((Y) \cap (X,Y)^2)$. Since $XY \in (Y) \cap (X,Y)^2$ and $X,Y \notin (X,Y)^2$, the element $X \in A$ is a zero divisor. We claim that $X$ is not in any minimal prime ideal of $A$. For, suppose $p \in \text{Spec } A$ contains $X$. We can write $p = q \setminus ((Y) \cap (X,Y)^2)$ for some $q \in \text{Spec } k[X,Y]$ such that $(Y) \cap (X,Y)^2 \subseteq q$. Note that $Y^2 \in q$, so $Y \in q$ as $q$ is prime. And since $p$ contains $X$, $q$ contains $X$. Thus $q$ contains the ideal $(X,Y)$, which is already maximal! So $q = (X,Y)$ and $p = (X,\overline{Y})$. But there is a chain of strict inclusions $(Y) \cap (X,Y)^2 \subseteq (Y) \subseteq (X,Y)$. So $(\overline{Y})$ is a prime ideal in $A$ (because $(Y)$ is prime in $k[X,Y]$) which is strictly contained in $p = (X,\overline{Y})$ thus $p$ is not prime.

**2.1.8.** Let $\varphi : A \to B$ be an integral ring homomorphism.

**(a)** Show that $\text{Spec } \varphi : \text{Spec } B \to \text{Spec } A$ maps a closed point to a closed point, and that any preimage of a closed point is a closed point.

Let $n$ be a maximal ideal of $B$. Then $\varphi$ induces an injective ring homomorphism

$$\psi : A/\varphi^{-1}(n) \hookrightarrow B/n.$$  

Since $\varphi$ is integral, so is $\psi$. But $B/n$ is a field and by the following lemma $A/\varphi^{-1}(n)$ is a field. Hence $\varphi^{-1}(n)$ is maximal in $A$. 

Lemma 3. If $B$ is an integral domain with a subring $A$ such that the inclusion $A \subseteq B$ is integral, then $A$ is a field if and only if $B$ is a field.

Proof. Suppose $B$ is a field and let $a \in A \setminus \{0\}$. Since $a^{-1} \in B$ is integral over $A$, there exists $b_0, \ldots, b_{n-1} \in A$ such that

\[
0 = b_0 + b_1 a^{-1} + \cdots + b_{n-1} (a^{-1})^{n-1} + (a^{-1})^n
= b_0 + b_1 a^{-1} + \cdots + b_{n-1} a^{-n+1} + a^{-n}
\]

so multiplying both sides by $a^{n-1}$ yields

\[
a^{-1} = -(b_0 + b_1 a^{-1} + \cdots + b_{n-1} a^{-n+1}) \in A.
\]

Suppose $A$ is a field and let $b \in B \setminus \{0\}$. Let $A[b]$ be the subring of $B$ generated by $A$ and $b$. Then there is a surjection $\epsilon : A[X] \to A[b]$ that sends $X$ to $b$. Since $b$ is integral over $A$, $\ker \epsilon \neq 0$. And since $A[b]$ is a domain $\ker \epsilon$ is a prime ideal in $A[X]$. But $A[X]$ is a PID, so $0 \neq \ker \epsilon$ must be a maximal ideal. Thus

\[
A[b] \cong A[X]/\ker \epsilon
\]

is a field. Thus $b$ has an inverse $b^{-1}$ in $A[b]$ and hence in $B$. \qed

The second part requires showing that given a prime ideal $q$ in $B$ such that $\varphi^{-1}(q)$ is maximal in $A$, then $q$ is maximal. Again using the injection $A/\varphi^{-1}(q) \hookrightarrow B/q$ and the lemma (in the other direction this time) we get that $B/q$ is a field hence $q$ is maximal.

(b) Let $p \in \text{Spec } A$. Show that the canonical homomorphism $A_p \to B \otimes_A A_p$ is integral.

We will show that in general for a multiplicative set $S$, the ring homomorphism $S^{-1}A \to S^{-1}B$ is integral, which is enough since $B \otimes_A S^{-1}A \cong S^{-1}B$ in a natural way. And indeed given $b \in B$ and $s \in S$, since $b$ is integral over $A$ we have

\[
a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots + a_{n-1} b^{n-1} + b^n = 0
\]

for some $a_0, \ldots, a_{n-1} \in A$. Therefore in $S^{-1}B$, we have

\[
0 = \frac{a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots + a_{n-1} b^{n-1} + b^n}{s^n}
= \frac{a_0}{s^n} + \frac{a_1}{s^{n-1}} \cdot \frac{b}{s} + \frac{a_2}{s^{n-2}} \cdot \left( \frac{b}{s} \right)^2 + \cdots + \frac{a_{n-1}}{s} \cdot \left( \frac{b}{s} \right)^{n-1} + \left( \frac{b}{s} \right)^n
\]

which shows that $b/s$ is integral over $S^{-1}A$.

(c) Let $T = \varphi(A \setminus p)$. Let us suppose that $\varphi$ is injective. Show that $T$ is a multiplicative subset of $B$, and that $B \otimes_A A_p = T^{-1}B \neq 0$. Deduce from this that $\text{Spec } \varphi$ is surjective if $\varphi$ is integral and injective.

Being a ring homomorphism, $\varphi$ maps the multiplicative subset $A \setminus p$ in $A$ to a multiplicative subset of $B$, which is called $T$ here. Since the functor $- \otimes_A A_p$ is exact, $A \otimes_A A_p = A_p$ injects in $B \otimes_A A_p$ because $A$ injects in $B$. As $A_p \neq 0$, we deduce that $B \otimes_A A_p \neq 0$. 
We want to show \( \text{Spec } \varphi \) is surjective. So let \( p \) be a prime ideal in \( A \) and write \( T = \varphi(A \setminus p) \) as above. Since \( T^{-1}B = B \otimes_A A_p \neq 0 \), \( T^{-1}B \) has a maximal ideal which is necessarily of the form \( T^{-1}q \) for a prime ideal \( q \) of \( B \) such that \( q \cap T = \emptyset \). Now \( \varphi_p : A_p \rightarrow B \otimes_A A_p \) is integral by (b), so by (a) \( \text{Spec } \varphi_p \) maps closed points to closed points. But there is only one closed point in \( \text{Spec } A_p \), so \( \text{Spec } \varphi_p(T^{-1}q) = A_p \). Applying \( \text{Spec} \) to the commutative diagram of rings

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_p & \xrightarrow{\varphi_p} & B \otimes_A A_p
\end{array}
\]

yields the following commutative diagram of spaces:

\[
\begin{array}{ccc}
\text{Spec } A & \xleftarrow{\text{Spec } \varphi} & \text{Spec } B \\
\uparrow & & \uparrow \\
\text{Spec } A_p & \xleftarrow{\text{Spec } \varphi_p} & \text{Spec}(B \otimes_A A_p)
\end{array}
\]

Chasing \( T^{-1}q \in \text{Spec}(B \otimes_A A_p) \) yields \( (\text{Spec } \varphi)(q) = p \). Observe that we started with an arbitrary \( p \in \text{Spec } A \) and found a \( q \in \text{Spec } B \) that maps to it. This means \( \text{Spec } \varphi \) is surjective.

2.2.13. Let \( f : X \rightarrow Y \) be a continuous map of topological spaces. Let \( \mathcal{F} \) be a sheaf on \( X \) and \( \mathcal{G} \) a sheaf on \( Y \).

(a) Show that there exist canonical morphisms of sheaves

\[
\mathcal{G} \rightarrow f_* f^{-1} \mathcal{G}, \quad f^{-1} f_* \mathcal{F} \rightarrow \mathcal{F}.
\]

Moreover, if \( f \) is a closed immersion, then the first morphism is surjective. If \( f \) is an open immersion, then the second morphism is an isomorphism.

(b) Show that there exists a canonical bijection

\[
\text{Hom}_{\text{Sh}}(\mathcal{G}, f_* \mathcal{F}) \simeq \text{Hom}_{\text{PrSh}}(f^{-1} \mathcal{G}, \mathcal{F}).
\]

Let \( \overline{\mathcal{F}} \) be the presheaf on \( X \) defined by

\[
U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V).
\]

So \( f^{-1} \mathcal{G} \) is the sheafification of \( \overline{\mathcal{F}} \). Therefore there is a canonical bijection

\[
\text{Hom}_{\text{Sh}}(f^{-1} \mathcal{G}, \mathcal{F}) \simeq \text{Hom}_{\text{PrSh}}(\overline{\mathcal{F}}, \mathcal{F})
\]

by the universal property of sheafification. It remains to show that there is a canonical bijection

\[
\text{Hom}_{\text{Sh}}(\mathcal{G}, f_* \mathcal{F}) \simeq \text{Hom}_{\text{PrSh}}(\overline{\mathcal{F}}, \mathcal{F}).
\]

So let \( \eta : \mathcal{G} \rightarrow f_*(\mathcal{F}) \) be a morphism of sheaves over \( Y \). Fix an open subset \( U \) in \( X \). Now for every \( V \) open in \( Y \) with \( f(U) \subseteq V \) we have an abelian group homomorphism

\[
\eta_V : \mathcal{G}(V) \rightarrow f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))
\]

and the restriction map \( \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U) \). Composing these yields a map

\[
\mathcal{G}(V) \rightarrow \mathcal{F}(U)
\]
for every such $V$. These maps are natural in $V$ because they are induced by $\eta_V$’s which form a natural transformation. Because of this compatibility the universal property of the direct limit yields a map

$$[\Xi(\eta)]_U : \overline{\mathcal{G}}(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U)$$

which is natural in $U$’s (respects restrictions), so we get a natural transformation (presheaf morphism)

$$\Xi(\eta) : \overline{\mathcal{G}} \to \mathcal{F}.$$  

We have just defined a map

$$\Xi : \text{Hom}_{\mathcal{Sht}_Y}(\mathcal{G}, f_*\mathcal{F}) \to \text{Hom}_{\mathcal{Prsh}_X}(\overline{\mathcal{G}}, \mathcal{F})$$

$$\eta \mapsto \Xi(\eta).$$

To get a map in the other direction, let $\theta : \overline{\mathcal{G}} \to \mathcal{F}$ be a presheaf morphism. Let $V$ be an open set in $Y$. Then we have an abelian group homomorphism

$$\theta_{f^{-1}(V)} : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)).$$

Note that by the direct limit property there is a map

$$\mathcal{G}(V) \to \lim_{W \supseteq f^{-1}(V)} \mathcal{G}(W) = \overline{\mathcal{G}}(f^{-1}(V))$$

since $V$ is an open set which contains $f(f^{-1}(V))$. Composing with $\theta_{f^{-1}(V)}$ gives a map

$$[\Lambda(\theta)]_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$$

which is natural in $V$ since $\theta$ is natural in $f^{-1}(V)$. Thus we get a natural transformation

$$\Lambda(\theta) : \mathcal{G} \to f_*\mathcal{F}.$$  

Thus we get a map

$$\Lambda : \text{Hom}_{\mathcal{Prsh}_X}(\overline{\mathcal{G}}, \mathcal{F}) \to \text{Hom}_{\mathcal{Sht}_Y}(\mathcal{G}, f_*\mathcal{F})$$

$$\theta \mapsto \Lambda(\theta).$$

Now let’s show that $\Xi$ and $\Lambda$ are mutually inverse. Start with $\eta \in \text{Hom}_{\mathcal{Sht}_Y}(\mathcal{G}, f_*\mathcal{F})$. Then for every open $V$ in $Y$, the map $[\Lambda(\Xi(\eta))]_V$ is given by composing $[\Xi(\eta)]_{f^{-1}(V)}$ with $\mathcal{G}(V) \to \overline{\mathcal{G}}(f^{-1}(V))$ which is $\eta_V$ by construction. So we obtain $\Lambda \circ \Xi = \text{id}$. The reverse composition is similarly the identity.

Now we have the bijection

$$\text{Hom}_{\mathcal{Sht}_Y}(\mathcal{G}, f_*\mathcal{F}) \simeq \text{Hom}_{\mathcal{Sht}_X}(f^{-1}\mathcal{G}, \mathcal{F})$$

for every sheaf $\mathcal{F}$ over $X$ and every sheaf $\mathcal{G}$ over $Y$. Then given a sheaf $\mathcal{F}$ over $X$, choosing $\mathcal{G} = f_*\mathcal{F}$ we get a bijection

$$\text{Hom}_{\mathcal{Sht}_Y}(f_*\mathcal{F}, f_*\mathcal{F}) \simeq \text{Hom}_{\mathcal{Sht}_X}(f^{-1}f_*\mathcal{F}, \mathcal{F})$$

The image of the identity on the left hand side is a morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

Similarly, given a sheaf $\mathcal{G}$ over $Y$, choosing $\mathcal{F} = f^{-1}\mathcal{G}$ we have a bijection

$$\text{Hom}_{\mathcal{Sht}_Y}(\mathcal{G}, f^{-1}f_*\mathcal{G}) \simeq \text{Hom}_{\mathcal{Sht}_X}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$$

so the image of the identity on the right hand side yields a morphism $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.  

Now suppose \( f \) is a closed immersion, so we can identify \( X \) as a closed subspace of \( Y \). Then if \( y \not\in X \) then \( V = Y \setminus X \) is an open neighborhood of \( y \) and
\[
f_* f^{-1} \mathcal{G}(V) = f^{-1} \mathcal{G}(V \cap X) = f^{-1} \mathcal{G}(\emptyset) = 0
\]
thus the stalk \( (f_* f^{-1} \mathcal{G})_y = 0 \). Hence the stalk map \( \mathcal{G}_y \to (f_* f^{-1} \mathcal{G})_y \) is surjective.

If \( f \) is an open immersion, then we may identify \( X \) as an open subspace of \( Y \). Then given \( x \in X \) we have \( (f^{-1} f_* F)_x = (f_* F)_x = F_x \) since the stalk can be computed locally on \( X \) as \( X \) is open.

2.3.2. Let \( U = \text{Spec} B \) be an affine open subscheme of \( X = \text{Spec} A \). Show that the restriction \( A \to B \) is a flat homomorphism.

Since there is a natural bijection between the ring homomorphisms from \( A \) to \( B \) and the ringed space morphisms from \( U \) to \( X \), the inclusion \( \text{Spec} B \to \text{Spec} A \) is equal to \( \text{Spec} \varphi \) for some ring homomorphism \( \varphi : A \to B \). We want to show that \( \varphi \) is flat.

Since \( U \) is an open subscheme of \( X \), for every point \( q \in U = \text{Spec} B \), the stalk map
\[
A_{\varphi^{-1}(q)} \to B_q
\]
\[
a \mapsto \varphi(a) \mapsto \varphi(s)
\]
is an isomorphism, in particular it is flat. Since this holds for every prime ideal \( q \) in \( B \), by Corollary 1.2.15 part (ii) \( \varphi \) is flat.

2.3.5. Let \( Y \) be a scheme that satisfies the conclusion of Proposition 3.25 for every affine scheme \( X \). Show that \( Y \) is affine.

Let \( \text{Sch} \) be the category of schemes. We have the global sections functor \( G : \text{Sch} \to \text{Ring} \) which is contravariant. We also have the spectrum functor \( \text{Spec} : \text{Ring} \to \text{Sch} \) which is adjoint with \( G \). We are assuming that there is a natural bijection
\[
\text{Hom}_{\text{Sch}}(X, Y) \simeq \text{Hom}_{\text{Ring}}(G(Y), G(X)).
\]
for every scheme \( X \). In particular, if \( X = \text{Spec} A \) is affine, we have
\[
\text{Hom}_{\text{Sch}}(X, Y) \simeq \text{Hom}_{\text{Ring}}(G(Y), G(X))
\]
\[
\simeq \text{Hom}_{\text{Ring}}(G(Y), A)
\]
\[
\simeq \text{Hom}_{\text{Sch}}(\text{Spec} A, \text{Spec} G(Y))
\]
\[
\simeq \text{Hom}_{\text{Sch}}(X, \text{Spec} G(Y)).
\]
By the proof of Proposition 3.25, we actually get natural bijections
\[
\text{Hom}_{\text{Sch}}(X, Y) \simeq \text{Hom}_{\text{Ring}}(G(Y), G(X)) \simeq \text{Hom}_{\text{Sch}}(X, \text{Spec} G(Y))
\]
for every scheme \( X \). This means that there is an isomorphism of contravariant functors
\[
\text{Hom}_{\text{Sch}}(\_, Y) \simeq \text{Hom}_{\text{Sch}}(\_, \text{Spec} G(Y)) : \text{Sch} \to \text{Set}.
\]
Hence by Yoneda's lemma \( Y \cong \text{Spec} G(Y) \) as a scheme so \( Y \) is affine.

2.3.16. Let \( X \) be a scheme. Show that \( X \) is locally Noetherian if and only if any affine open subscheme of \( X \) is Noetherian.

First we show that “locally Noetherian” is a really local condition:
Lemma 4. Let $X$ be a locally Noetherian scheme. Then the set of open Noetherian subschemes form a base for the topology of $X$.

Proof. Let $x \in X$ and $U$ be a neighborhood of $x$ in $X$. By assumption, $x$ has a Noetherian open neighborhood $V$ in $X$. So by Proposition 3.46 part (a), $U \cap V$ is a Noetherian neighborhood of $x$ which is contained in $U$. This verifies the condition for being a base. □

It immediately follows from this lemma that an open subset of a locally Noetherian scheme is also locally Noetherian.

To answer the question, the “if” direction is clear because $X$ can be covered by affine schemes by definition; since we are assuming each of those are Noetherian we obtain that $X$ is locally Noetherian.

For the converse, suppose that $X$ is locally Noetherian. Let $Y$ be an affine open subscheme of $X$. Then $Y$ is also locally Noetherian. So $Y$ has an open covering of Noetherian schemes, but since $Y$ is quasi-compact $Y$ has a finite open covering of Noetherian schemes, say $Y = \bigcup_{i=1}^{n} U_i$ where $U_i$ is open and Noetherian. But then each $U_i$ can be covered by finitely many affine open Noetherian schemes, hence $Y$ is covered by finitely many affine open Noetherian schemes. By definition, then $Y$ is Noetherian.

2.4.8. Let $X$ be a quasi-compact scheme. Show that $X$ contains a closed point.

Let’s define a topological space to be cool if it is nonempty and every nonempty closed subset contains a closed point.

Lemma 5. Let $X$ be a topological space which has a finite open covering of cool subsets. Then $X$ contains a closed point.

Proof. We show by induction on $n$ that if $X$ can be written as a union of $n$ cool and open subsets then $X$ has a closed point. The basis case is easy since then $X$ is cool and hence the closed subset $X$ contains a closed point.

Now assume that the claim is true for $n - 1$. Suppose $X = \bigcup_{j=1}^{n} U_j$ where each $U_j$ is cool and open. By induction hypothesis, the subspace $X_1 = \bigcup_{j=2}^{n} U_j$ contains a closed point $x$. Note that this means $\{x\} \cap X_1 = \{x\}$. Consider the closed subset $\{x\} \cap U_1$ of $U_1$. There are two cases:

- $\{x\} \cap U_1 = \emptyset$. Then $\{x\} = (\{x\} \cap X_1) \cup (\{x\} \cap U_1) = \{x\}$ hence $x$ is a closed point of $X$.
- $\{x\} \cap U_1 \neq \emptyset$. Then by the coolness of $U_1$ its closed subset $\{x\} \cap U_1$ contains a closed point $x_1$ of $U_1$. So we have $x_1 \in \{x\}$ and $\{x_1\} \cap U_1 = \{x_1\}$. We want to show that $x_1$ is a closed point of $X$. Observe that $\{x_1\} \cap X_1 \subseteq \{x\} \cap X_1 = \{x\}$.

Suppose $x \in \overline{\{x_1\}}$. Then $x \notin U_1$ so $X \setminus U_1$ is a closed subset of $X$ containing $x$. This forces $x_1 \in X \setminus U_1$; a contradiction. Thus $x \notin \{x_1\}$ and hence $\{x_1\} \cap X_1 = \emptyset$. Thus $\{x_1\} = (\{x_1\} \cap X_1) \cup (\{x_1\} \cap U_1) = \{x_1\}$.
which means \( x_1 \) is a closed point in \( X \).

\[ \square \]

Note that affine schemes are cool topological spaces because every nonempty closed subset \( V(I) \) in \( \text{Spec} \ A \) contains a maximal ideal. Thus quasi-compact schemes, which can be covered by finitely many affine schemes contain a closed point.

**2.4.12.** Let \( B \) be a graded ring. Let \( Y \) be a reduced closed subscheme of \( \text{Proj} \ B \). Show that there exists a homogenous ideal \( I \) of \( B \) such that \( Y \cong \text{Proj} \ B/I \).

As a topological subspace \( Y \) can be identified with \( V_+(I) \) for a homogenous ideal \( I \) of \( B \) because that’s what the closed subspaces of \( \text{Proj} \ B \) look like.

**Lemma 6.** With the notation above, \( \sqrt{I} \) is also a homogenous ideal.

**Proof.** Suppose that \( p \) is a prime ideal containing \( I \). Then for every \( d \geq 0 \)

\[
I = \oplus_{d \geq 0} (I \cap B_d) \subseteq \oplus_{d \geq 0} (p \cap B_d) = p^h.
\]

and we know by Lemma 3.35 that \( p^h \) is prime. So we have shown that \( p \in V(I) \) implies \( p^h \in V_+(I) \). Hence

\[
\bigcap_{p \in V(I)} p \supseteq \bigcap_{p \in V(I)} p^h \supseteq \bigcap_{q \in V_+(I)} q
\]

but since \( V_+(I) \subseteq V(I) \) the reverse containment is trivial; hence

\[
\sqrt{I} = \bigcap_{p \in V(I)} p = \bigcap_{q \in V_+(I)} q
\]

is homogenous. \( \square \)

Now we have that \( \sqrt{I} \) is homogenous and moreover \( V_+(I) = V_+(\sqrt{I}) \). Thus we may assume \( I = \sqrt{I} \).

The ring \( B/I \) is reduced since \( \sqrt{B/I} = \sqrt{I}/I = 0 \). Thus the scheme \( \text{Spec} \ B/I \) is reduced. So in particular for every homogenous \( f \in A := B/I \), \( \mathcal{O}_{\text{Spec} \ A}(D(f)) = A_f \) is a reduced ring. This implies that the subring \( A_f \) of \( A_f \) is also reduced. Since \( A_f = \mathcal{O}_{\text{Proj} \ A}(D_+(f)) \) and \( \{D_+(f) : f \text{ homogenous}\} \) forms an affine open covering of \( \text{Proj} \ A \), by Proposition 4.2 part (b) \( \text{Proj} \ A = \text{Proj} B/I \) is a reduced scheme which can be identified as a closed subscheme of \( \text{Proj} B \) whose underlying topological space is \( V_+(I) \).

But by Proposition 4.2 part (d), there is a unique structure of a reduced closed subscheme on \( V_+(I) \). Since \( \text{Proj} B/I \) and \( Y \) are both such subschemes, we get \( Y \cong \text{Proj} B/I \).