Let $Y$ be a Noetherian scheme. Show that any $Y$-scheme $X$ of finite type is Noetherian. Moreover, if $Y$ is of finite dimension, then so is $X$.

Write $f : X \to Y$ for the structure morphism which is assumed to be of finite type. To show that $X$ is Noetherian, it suffices to show that it is quasi-compact and locally Noetherian.

Since $Y$ is quasi-compact, it can be covered by finitely many affine open subsets, say $V_1, \ldots, V_n$. Then $f^{-1}(V_j)$’s cover $X$. Now $f$ is quasi-compact, so each $f^{-1}(V_j)$ is quasi-compact and hence their finite union $X$ is quasi-compact.

To show that every point in $X$ has a Noetherian neighborhood, let $x \in X$. Since $Y$ is locally Noetherian, $f(x)$ is contained in an affine open subset $V = \text{Spec } B$ of $Y$ such that $B$ is Noetherian. Let $U = \text{Spec } A$ be an affine neighborhood of $x$ such that $U \subseteq f^{-1}(V)$. Then since $f$ is of finite-type, $A$ is a finitely generated $B$-algebra, hence $A$ is Noetherian.

Moreover, in the above notation, if $Y$ has finite dimension, then $\dim B < \infty$ and so by Corollary 2.5.17 the finitely generated $B$-algebra $A$ has finite dimension. So every $x \in X$ has a neighborhood with finite dimension. Since $X$ is quasi-compact, $X$ itself has finite dimension.

Show that any open immersion into a localy Noetherian scheme is a morphism of finite type.

Let $f : X \to Y$ be an open immersion where $Y$ is locally Noetherian. Let $V$ be an affine open subscheme of $Y$. Then by Exercise 2.3.16 $V$ is Noetherian. Let $U = f^{-1}(V)$, then clearly the restriction $f|_U : U \to V$ is also an open immersion; hence by Proposition 2.3.46(a) $U$ is Noetherian. By definition, $U$ is then a finite union of affine open (hence quasi-compact) subschemes, therefore $U = f^{-1}(V)$ is quasi-compact. This shows that $f$ is quasi-compact.

Let $\{V_i\}$ be an open covering of $Y$ by affine open subsets. Via $f$, we can identify $X$ as an open subset of $Y$. For each $i$, $f^{-1}(V_i)$ is quasi-compact by above. Identifying $X$ as an open subscheme of $Y$, $f^{-1}(V_i) = V_i \cap X$ is an open subset of the affine scheme $V_i$ and hence by quasi-compactness can be covered by finitely many principal open sets $U_{ij}$ of $V_i$. We can make the identifications $V_i = \text{Spec } A_i$ for some ring $A_i$ and $U_{ij} = D(g_{ij})$ where $g_{ij} \in A_i$. Observe that $\mathcal{O}_Y(V_i) = A_i$ and $\mathcal{O}_X(U_{ij}) = \mathcal{O}_Y(U_{ij}) = \mathcal{O}_{V_i}(U_{ij}) = (A_i)_{g_{ij}}$ such that the induced map $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_{ij})$ is the localization $A_i \to (A_i)_{g_{ij}}$.

In general, if $A$ is a ring and $g \in A$, $A[X]$ surjects on $A_g$ by the $A$-algebra homomorphism $X \mapsto 1/g$, so the localization $A \to A_g$ makes $A_g$ into a finitely generated $A$-algebra. Thus $\mathcal{O}_X(U_{ij})$ is a finitely generated algebra over $\mathcal{O}_Y(V_i)$ for every $j$. Hence by Proposition 3.2.2 $f$ is of finite type.
3.2.3. An *immersion* of schemes is a morphism which is an open immersion followed by a closed immersion.

(a) Let \( f : X \to Y \) be an immersion. Show that it can be decomposed into a closed immersion followed by an open immersion.

(b) Show that the converse of (a) is true if \( f \) is moreover quasi-compact (e.g. if \( Y \) is locally Noetherian). Use the scheme-theoretic closure of \( f \) (Exercise 2.3.17).

(c) Let \( f : X \to Y, \, g : Y \to Z \) be two immersions with \( g \) quasi-compact. Show that \( g \circ f \) is an immersion.

(a) \( Y \) has a closed subscheme \( C \) such that there is an open immersion \( h : X \to C \) such that when composed with the closed immersion \( i : C \hookrightarrow Y \) we get \( f \), that is, \( f = i \circ h \). Identify \( X \) as an open subspace of \( C \) via \( f \). Then \( Y \) has an open subset \( V \) such that \( X = C \cap V \). Since \( f(X) \subseteq V \) and \( V \) is open, \( f \) restricts to a scheme morphism \( g : X \to V \), that is, if we write \( j : V \hookrightarrow Y \) for the open immersion, then \( f = j \circ g \).

Since \( X \) is identified with the closed subspace \( C \cap V \) of \( V \) via \( f \), the morphism \( g \) is a closed immersion of schemes. So \( j \circ g \) is a decomposition of the desired form.

(b) LATER

(c) Write \( g = a \circ b \) and \( f = c \circ d \) where \( b, d \) are open immersions and \( a, c \) are closed immersions. Since \( g \) is quasi-compact and \( a \) is an open immersion \( b \) is also quasi-compact. Also \( c \), being an open immersion is quasi-compact. Thus the composite \( b \circ c \) is quasi-compact. Now by (b), \( b \circ c = b' \circ c' \) where \( b' \) is an open immersion and \( c' \) is a closed immersion. Thus

\[
g \circ f = (a \circ b') \circ (c' \circ d)
\]

3.2.4. Let \( X, Y \) be schemes over a locally Noetherian scheme \( S \), with \( Y \) of finite type over \( S \). Let \( x \in X \). Show that for any morphism of \( S \)-schemes \( f_x : \text{Spec} \, \mathcal{O}_{X,x} \to Y \), there exist an open subset \( U \ni x \) of \( X \) and a morphism of \( S \)-schemes \( f : U \to Y \) such that \( f_x = f \circ i_x \), where \( i_x : \text{Spec} \, \mathcal{O}_{X,x} \to U \) is the canonical morphism (in other words, the morphism \( f_x \) extends to an open neighborhood of \( x \)).

We have a commutative diagram of schemes

\[
\begin{array}{ccc}
\text{Spec} \, \mathcal{O}_{X,x} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
& \text{Spec} \, \mathcal{O}_S & \to \end{array}
\]

where the diagonal morphism is given by the structure morphism \( h : X \to S \) Consider the image of \( x \) in \( S \) under the diagonal arrow and find an affine open neighborhood \( V \) around it. Then \( h^{-1}(V) \) is an open subset in \( X \) containing \( x \) so it contains an affine neighborhood \( U \) of \( x \) in \( X \). Then since \( \mathcal{O}_X(U) \) maps to \( \mathcal{O}_{X,x} \), \( \text{Spec} \, \mathcal{O}_{X,x} \to S \) factors through the restriction \( U \to V \). That is, the diagonal arrow maps to an affine open subset of \( S \). Then making the same argument for \( f_x \) by taking an affine open \( W \) in \( Y \) which maps to \( V \) and shrinking \( U \) if necessary, we reduce to the case where all \( X, Y \) and \( S \) are affine.
Writing $X = \text{Spec } A$, $Y = \text{Spec } B$ and $S = \text{Spec } C$ and $x = p \in \text{Spec } A$, the commutative diagram of (now affine) schemes above correspond to the following commutative diagram of rings:

$$
\begin{array}{ccc}
A_p & \rightarrow & B \\
& \searrow & \\
& C & \\
\end{array}
$$

In addition to this, there is a ring homomorphism from $C$ to $A$ which is compatible with the above picture. So more compactly, both $A$ and $B$ are $C$-algebras and $\varphi : B \rightarrow A_p$ is a $C$-algebra homomorphism. We want to find an $f \in A \setminus p$ such that $\varphi$ factors through the canonical map $A_f \rightarrow A_p$; because on the scheme level this amounts to extending the map from $\text{Spec } \mathcal{O}_{X,x}$ to the principal open set $D(f)$.

Note that since $C$ came from an affine open subset of a locally Noetherian scheme, we may assume $C$ is Noetherian. And the finite type assumption ensures that $B$ is a finitely generated $C$-algebra. So there exists $n \in \mathbb{N}$ such that there is a surjective $C$-algebra homomorphism

$$
\pi : C[T_1, \ldots, T_n] \rightarrow B
$$
say with kernel $I$. Because $C$ is noetherian, so is $C[T_1, \ldots, T_n]$ (Hilbert basis theorem) therefore $I$ is finitely generated. So $I = (g_1, \ldots, g_r)$ for some polynomials $g_1, \ldots, g_r \in C[T_1, \ldots, T_n]$. Write $\pi(T_i) = b_i \in B$ and $\varphi(b_i) = \frac{a_i}{s_i} \in A_p$ (so $a_i \in A$ and $s_i \in A \setminus p$) for each $i = 1, \ldots, n$.

Fix $j \in \{1, \ldots, m\}$. Now we know that $g_j(b_1, \ldots, b_n) = 0$ in $B$, therefore $g_j(\frac{a_1}{s_1}, \ldots, \frac{a_n}{s_n}) = 0$ in $A_p$. Clearing the denominators in the latter equation, we get a polynomial $\tilde{g}_j \in A[T_1, \ldots, T_n]$ such that

$$
\frac{\tilde{g}_j(a_1, \ldots, a_n)}{1} = 0 \in A_p
$$

hence there exists $t_j \in A \setminus p$ such that

$$
t_j \tilde{g}_j(a_1, \ldots, a_n) = 0
$$
in $A$ (we may take $\tilde{g}_j = (\prod_{i=1}^n s_i) g_j$).

Now let

$$
f = \prod_{i=1}^n s_i \cdot \prod_{j=1}^m t_j \in A \setminus p.
$$

Since $f$ is invertible in $A_f$, every $s_i$ is invertible in $A_f$. So we can define a $C$-algebra homomorphism

$$
\psi : C[T_1, \ldots, T_n] \rightarrow A_f \\
T_i \mapsto s_i^{-1} a_i.
$$

Now since every $t_j$ in $A_f$, we have

$$
\frac{\tilde{g}_j(a_1, \ldots, a_n)}{1} = 0 \in A_f
$$
so by decomposing back we get
\[ g_j(s_1^{-1}a_1, \ldots, s_n^{-1}a_n) = 0. \]

Therefore the polynomial \( g_j \in C[T_1, \ldots, T_n] \) lies in the kernel of \( \psi \). Since this is true for every \( j = 1, \ldots, m \) we get \( I \subseteq \ker \psi \). Thus \( \psi \) uniquely factors through \( B \), that is, we get a \( C \)-algebra homomorphism \( B \to A_f \) which by construction makes

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi} & A_p \\
\downarrow & & \downarrow \\
A_f & & 
\end{array}
\]

commute. This is what we wanted.

3.2.5. Let \( S \) be a locally Noetherian scheme. Let \( X, Y \) be \( S \)-schemes of finite type. Let us fix \( s \in S \). Let \( \varphi : X \times_S \text{Spec} \mathcal{O}_{S,s} \to Y \times_S \text{Spec} \mathcal{O}_{S,s} \) be a morphism of \( S \)-schemes. Show that there exist an open set \( U \ni s \) and a morphism of \( S \)-schemes \( f : X \times_S U \to Y \times_S U \) such that \( \varphi \) is obtained from \( f \) by the base change \( \text{Spec} \mathcal{O}_{S,s} \to U \). If \( \varphi \) is an isomorphism, show that there exists such an \( f \) which is moreover an isomorphism.

We first solve the problem for for \( S = \text{Spec} A \), \( X = \text{Spec} B \), \( Y = \text{Spec} C \) are all affine. By the assumptions, \( A \) is Noetherian and \( B, C \) are finitely generated \( A \)-algebras. Write \( s = p \). So we have an \( A \)-algebra map \( \varphi : B_p \to C_p \). Let's write \( \psi : B \to C_p \) for the composition of \( \varphi \) with the localization \( B \to B_p \). Say \( b_1, \ldots, b_n \) generate \( B \) as an \( A \)-algebra. Write

\[ \psi(b_j) = \frac{c_j}{f_j} \]

for each \( j \) where \( c_j \in C \) and \( f_j \in A - p \). Let \( f = \prod_{j=1}^n f_j \in A - p \). So \( \psi \) factors through \( B_f \), say via \( \sigma : B_f \to C_p \).

3.2.6. Let \( f : X \to Y \) be a morphism of integral schemes. Let \( \xi \) be the generic point of \( X \). We say that \( f \) is birational if \( f^\#_\xi : K(Y) \to K(X) \) is an isomorphism. Let us suppose that \( X \) and \( Y \) are of finite type over a scheme \( S \), and that \( f \) is a morphism of \( S \)-schemes. Show that \( f \) is birational if and only if there exists a non-empty open subset \( U \) of \( X \) and an open subset \( V \) of \( Y \) such that \( f \) induces an isomorphism from \( U \) onto \( V \).

3.2.7. Let \( k \) be a field and \( \bar{k} \) the algebraic closure of \( k \). Let \( \bar{X}, \bar{Y} \) be algebraic varieties over \( \bar{k} \), \( \bar{f} \) a morphism from \( \bar{X} \) to \( \bar{Y} \). Show that there exist a finite extension \( K \) of \( k \), algebraic varieties \( X, Y \) over \( K \), and a morphism \( f : X \to Y \), such that \( X_\bar{k} = \bar{X}, Y_\bar{k} = \bar{Y}, \) and \( \bar{f} = f_\bar{k} \).

3.2.8. Let \( f : X \to Y \) be a morphism of schemes. We say that \( f \) has finite fibers if \( f^{-1}(y) \) is a finite set for every \( y \in Y \). We say that \( f \) is quasi-finite ([41], II.6.2.3 and ErrIII, 20) if, moreover, \( \mathcal{O}_{X,x} \) is finite over \( k(y) \) for every \( x \in X_y \). Show that a morphism of finite type with finite fibers is quasi-finite. Give an example of a morphism with finite fibers which is not quasi-finite.
3.2.16. Let $A$ be an integral domain with field of fractions $K$, and let $B$ be a finitely generated $A$-algebra. We suppose that $B \otimes_A K$ is finite over $K$.

(a) For any $b \in B$, show that there exists a $g \in A \setminus \{0\}$ such that $gb$ is integral over $A$.

(b) Show that there exists an $h \in A \setminus \{0\}$ such that $B_h$ is finite over $A_h$.

(a) Write $S = A \setminus \{0\}$. So $S^{-1}B$ is finite over $K = S^{-1}A$. Then for every $b \in B$, $b/1 \in S^{-1}B$ is integral over $S^{-1}A$ so there exists $a_0, a_1, \ldots, a_{n-1} \in A$ and $s_0, s_1, \ldots, s_{n-1} \in S$ such that

$$
\left( \frac{b}{1} \right)^n + \frac{a_{n-1}}{s_{n-1}} \left( \frac{b}{1} \right)^{n-1} + \cdots + \frac{a_1}{s_1} \frac{b}{1} + \frac{a_0}{s_0} = 0.
$$

Write $t = \prod_{k=0}^{n-1} s_k \in S$. Then multiplying the above equation by $t^n$ we obtain

$$
\left( \frac{tb}{1} \right)^n + \frac{a_{n-1}t}{s_{n-1}} \left( \frac{tb}{1} \right)^{n-1} + \cdots + \frac{a_1 t^{n-1}}{s_1} \left( \frac{tb}{1} \right) + \frac{a_0 t^n}{s_0} = 0.
$$

Since $s_k \mid t$ for every $k = 0, \ldots, n - 1$, we can write $c_k = \frac{a_k t^{n-k}}{s_k} \in A$. So we have

$$
\frac{(tb)^n + c_{n-1}(tb)^{n-1} + \cdots + c_1(tb) + c_0}{1} = 0
$$
in $S^{-1}B$. So there exists $s \in S$ such that

$$
s(tb)^n + sc_{n-1}(tb)^{n-1} + \cdots + sc_1(tb) + sc_0 = 0
$$
in $B$. Then multiplying by $s^{n-1}$, we get

$$(stb)^n + s^2 c_{n-1}(stb)^{n-1} + \cdots + s^{n-2} c_1(stb) + s^n c_0 = 0.$$ 

Write $g = st \in S$. Then the monic polynomial

$$p(X) = X^n + s^2 c_{n-1}X^{n-1} + \cdots + s^{n-2} c_1 X + s^n c_0 \in A[X]$$
is satisfied by $gb$.

(b) Suppose $B$ is generated by $b_1, \ldots, b_n \in B$ as an $A$-algebra. By (a), there exists $g_1, \ldots, g_m \in A \setminus \{0\}$ such that $g_j b_j$ is integral over $A$. The elements $g_j \cdot 1_B \in B$ are trivially integral over $A$, therefore if we write $h = \prod_{j=1}^m g_j \in A \setminus \{0\}$, the element $hb_j \in B$ is integral over $A$ for every $j$. Hence the element $\frac{hb_j}{1} \in B_h$ is integral over $A_h$.

But $h$ is a unit in $A_h$, so multiplying by $1/h$ we obtain that each $\frac{b_j}{1} \in B_h$ is integral over $A_h$. But these elements generate $B_h$ over $A_h$; thus $B_h$ is finite over $A_h$ (recall that finitely generated + integral implies finite).

3.2.17. Let $Y$ be an irreducible scheme with generic point $\xi$, and let $f : X \to Y$ be a dominant morphism of finite type (Exercise 2.4.11).

(a) Let $x$ be a closed point of $X_{\xi}$. Let $Z$ be the reduced subscheme $\overline{\{x\}}$ of $X$.

Show that there exists a dense open subset $V$ of $Y$ such that $f^{-1}(V) \cap Z \to V$ is surjective (use Exercises 2.16 and 2.1.8).

(b) Deduce from this that $f(X)$ contains a dense open subset of $Y$. 
(c) Show that \( f(X) \) is a constructible subset of \( Y \), that is \( f(X) \) is a finite disjoint union of sets \( Z_i \), with each \( Z_i \) the intersection of an open subset and a closed subset (see [43], Exercise II.3.18).

### 3.2.21
Let \( B \) be a finitely generated algebra over a ring \( A \). Show that there exist a subring \( A_0 \) of \( A \) that is finitely generated over \( Z \) and a finitely generated \( A_0 \)-algebra \( B_0 \) such that \( B = B_0 \otimes_{A_0} A \). If \( B \) is homogenous, we can take \( B_0 \) as homogenous.

**Remark.** We can show, more generally, that if \( X \) is a scheme of finite type over \( A \) with \( A \) Noetherian, then there exist \( A_0 \) as above and an \( A_0 \)-scheme of finite type \( X_0 \) such that \( X = X_0 \otimes_{\text{Spec} A_0} \text{Spec} A \). See [41], IV.8.8.2 (ii).

### 3.3.5
Let \( X \to S \) be a proper morphism. Let \( f : X \to Y \) be an open immersion of \( S \)-schemes. Let us suppose that \( Y \) is separated over \( S \) and connected. Show that \( f \) is an isomorphism.

It suffices to show that \( f \) is surjective, since by the definition of open immersion \( f(X) \) is an open subset of \( Y \) and \( f \) factors as

\[
(X, \mathcal{O}_X) \xrightarrow{\sim} (f(X), \mathcal{O}_Y|_{f(X)}) \xrightarrow{\sim} (Y, \mathcal{O}_Y)
\]

Write \( g : Y \to S \) for the structure morphism of \( Y \). We are given that \( g \) is separated and \( g \circ f \) is proper. Then by Proposition 3.16(e), \( f \) is proper. In particular, \( f \) is a closed map. Thus \( f(X) \) is both open and closed in \( Y \). So \( f(X) \) is a nonempty (\( X \) should be nonempty for the question to make sense) subset which is both open and closed in \( Y \). Since \( Y \) is connected, this forces \( f(X) = Y \).

### 3.3.9
Let \( S \) be a locally Noetherian scheme, \( s \in S \). Let \( Y \) be a scheme of finite type over \( \text{Spec} \, \mathcal{O}_{S,s} \). Show that there exist an open neighborhood \( U \) of \( s \) and a scheme of finite type \( X \to U \) such that \( Y = X \times_U \text{Spec} \, \mathcal{O}_{S,s} \). Moreover, if \( Y \) is separated, we can take \( X \to U \) separated.

### 3.3.11
Let \( X \) be a \( Y \)-scheme.

(a) Show that for any \( Y \)-scheme \( Z \), \( X_{\text{red}} \times_Y Z \to X \times_Y Z \) is a closed immersion and a homeomorphism on the underlying topological spaces. Deduce from this that \( X \to Y \) separated \( \iff X_{\text{red}} \to Y \) separated \( \iff X_{\text{red}} \to Y_{\text{red}} \) separated.

(b) Let \( \{F_i\}_i \) be a finite number of closed subsets of \( X \) such that \( X = \bigcup_i F_i \). Let us endow \( F_i \) with the structure of reduced closed subschemes. Show that \( X \to Y \) is separated if and only if \( F_i \to Y \) is separated for every \( i \).

(c) Let us suppose \( X \) is of finite type over \( Y \). Show that \( X \to Y \) proper \( \iff X_{\text{red}} \to Y \) proper \( \iff X_{\text{red}} \to Y_{\text{red}} \) proper. Also show that (b) is true for proper morphisms (by supposing \( X \to Y \) of finite type).

### 3.3.12
Let \( f : X \to Y \) be a surjective morphism. We suppose that \( Y \) is connected and that all of the fibers \( X_y \) are connected.

(a) Show that if \( X \) is proper over \( Y \), then \( X \) is connected.

(b) Study the example \( \text{Spec} \, k[T_1, T_2]/T_1(T_1T_2 - 1) \to \text{Spec} \, k[T_1] \), where \( k \) is a field. Deduce from this that (a) is false if \( f \) is not proper.
(a) For the sake of contradiction, suppose that $X$ is not connected. Then there exist disjoint nonempty closed subsets $C$ and $D$ of $X$ such that $X = C \cup D$. Since $f$ is proper, in particular $f$ is a closed map. So $Y = f(X)$ is equal to the union of the nonempty closed subsets $f(C)$ and $f(D)$. As $Y$ is connected, $f(C) \cap f(D) \neq \emptyset$. Pick a $y$ from this intersection, so there exist $x \in C$ and $t \in D$ such that $f(x) = f(t) = y$. So $x, y$ are both in $f^{-1}(y)$.

Now we have $x \in f^{-1}(y) \cap C$ and $t \in f^{-1}(y) \cap D$. So $f^{-1}(y) \cap C$ and $f^{-1}(y) \cap D$ are nonempty disjoint closed subsets of $f^{-1}(Y)$ and they cover $f^{-1}(y)$. Thus $f^{-1}(y)$ is not connected. But by Proposition 3.1.16, $f^{-1}(y)$ is homeomorphic with the fiber $X_y$, which is connected by assumption; a contradiction.

(b) Let $I$ and $J$ be the ideals of the ring $A = k[T_1, T_2]$ generated by $T_1$ and $T_1T_2 - 1$, respectively. Since 

$$1 = T_2 \cdot T_1 - (T_1T_2 - 1),$$

$I$ and $J$ are coprime, that is, $I + J = A$. Thus by the Chinese remainder we have $A/IJ \cong A/I \times A/J$.

Write $B = k[T]$. Consider the $k$-algebra homomorphism $B \to A$ given by $T \mapsto T_1$. Composing with the projections $A \to A/I$ and $A \to A/J$, we get ring homomorphisms $f : B \to A/I$ and $g : B \to A/J$. Note that $A/I$ is isomorphic to $B$ and $A/J$ is isomorphic to the localization $B_T$. Under these identifications, we get $f$ to be given by the map $T \mapsto 0$ and $g$ to be the localization map $B \to B_T$ ($T$ gets sent to $T_1$ which is formally inverted with the inverse $T_2$). Let’s consider two cases for a prime ideal in $p$ in $B$:

- $T \in p$. On one hand, there is no prime in $B_T$ whose inverse image under $g$ is $p$. On the other hand, since $(T)$ is maximal in $B$, we get $p = (T)$. Observe that for every $q \in \text{Spec } B$, we have $p = f^{-1}(0) \subseteq f^{-1}(q)$. But $p$ is maximal and $f^{-1}(q)$ is prime in $A$, so $p = f^{-1}(q)$.

- $T \notin p$. Then there is a unique $q \in \text{Spec } B_T$ such that $g^{-1}(q) = p$, namely $q = p_T$. However, by what we have seen above, there is no prime ideal in $B$ whose inverse image under $f$ is $p$.

Thus if we consider the ring homomorphism 

$$h : B \to A/I \times A/J$$

$$b \mapsto (f(b), g(b)),$$

then 

$$\text{Spec } h : \text{Spec } A/I \sqcup \text{Spec } A/J \to \text{Spec } B$$

$$q \mapsto \begin{cases} f^{-1}(q) & \text{if } q \in \text{Spec } A/I \\ g^{-1}(q) & \text{if } q \in \text{Spec } A/J \end{cases}$$

Let $p \in \text{Spec } B$. By our analysis above, if $T \notin p$ there is a unique $q \in \text{Spec } A/IJ$ such that $(\text{Spec } h)(q) = p$, that is, $(\text{Spec } h)^{-1}(p)$ is a singleton, hence connected. And if $T \in p$, $(\text{Spec } h)^{-1}(p) = \text{Spec } B = A_k^1$ is irreducible since $B$ is a domain, and in particular connected. But Spec $A/IJ = \text{Spec } A/I \sqcup \text{Spec } A/J$ is not connected.

3.3.13. (Rational maps) Let $X, Y$ be schemes over $S$, which are integral, with $Y \to S$ separated. A rational map from $X$ to $Y$, denoted by $X \dashrightarrow Y$, is an
equivalence class of morphisms of $S$-schemes from a non-empty open subscheme of $X$ to $Y$. Two such morphisms $U \to Y$, $V \to Y$ are called equivalent if they coincide on $U \cap V$. Let us fix a rational map $f : X \dasharrow Y$.

(a) Show that in every equivalence class, there exists a unique element $f : U \to Y$ such that $U$ is maximal for the inclusion relation, and that every element $g : V \to Y$ of the class verifies $g = f|_V$. We call $U$ the domain of definition of $X \dasharrow Y$. We then denote the rational map associated to $f$ by $f : X \dasharrow Y$.
(b) Let us suppose that $f$ is dominant, that $X, Y$ are of finite type over $S$, and that $S$ is locally Noetherian. Let $x \in X$. Show that $f$ is defined at $x$ (i.e., $x \in U$) if and only if there exists a $y \in Y$ such that the image of $\mathcal{O}_{Y,y}$ under $K(Y) \to K(X)$ is a local ring dominated by $\mathcal{O}_{X,x}$.
(c) Let $\Gamma_f$ be the closure in $X \times_S Y$ of the graph of $f : U \to Y$. We call it the graph of the rational map $f$. We endow $\Gamma_f$ with the structure of a reduced closed subscheme. Show that $\Gamma_f$ is integral and that the projection $p : X \times_S Y \to X$ induces a birational morphism from $\Gamma_f$ to $X$.
(d) Show that there exist a birational morphism $g : Z \to X$ and a morphism $\tilde{f} : Z \to Y$ such that $\tilde{f} = f \circ g$ on $g^{-1}(U)$ (we say that we have eliminated the indetermination of $X \dasharrow Y$). Moreover, if $Y$ is proper (resp. projective) over $S$, we can choose $g$ proper (resp. projective).

3.3.14. Let $X, Y$ be integral separated schemes over a Noetherian scheme $S$. Let $f : X \dasharrow Y$ be a birational map (i.e. $f$ comes from an isomorphism from a non-empty open subset $U \subseteq X$ onto an open subset of $Y$). Show that $f$ is defined everywhere if and only if $\Gamma_f \to X$ is an isomorphism.

3.3.15. We say that a morphism of schemes $f : X \to Y$ is finite (resp. integral) if for every affine open subset $V$ of $Y$, $f^{-1}(V)$ is affine and $\mathcal{O}_X(f^{-1}(V))$ is finite (resp. integral) over $\mathcal{O}_Y(V)$. Let $f : X \to Y$ be a morphism such that there exists an affine open covering $Y = \bigcup_i Y_i$ with $f^{-1}(Y_i)$ affine and $\mathcal{O}_X(f^{-1}(Y_i))$ finite over $\mathcal{O}_Y(Y_i)$. We want to show that $f$ is finite.

(a) Show that we can reduce to the case when $Y$ is affine and that $Y_i = D(h_i)$ a principal open subset.
(b) Show that $X$ then verifies condition (3.2), Subsection 2.3.1. Deduce from this that the canonical morphism $X \to \text{Spec} \mathcal{O}_X(X)$ is an isomorphism (hence $X$ is affine).
(c) Show that $\mathcal{O}_X(X)$ is finite over $\mathcal{O}_Y(Y)$.

3.3.16. Present a statement analogous to that of Exercise 3.15 by replacing finite morphism with integral morphism.

3.3.17. Show that the following properties are true.

(a) Any finite morphism is of finite type and quasi-finite.
(b) The class of finite morphisms verifies hypotheses (1)-(3) of Lemma 3.15.
(c) Let $\rho : A \to B$ be a finite ring homomorphism. Then for any ideal $I$ of $B$, we have $(\text{Spec} \rho)(V(I)) = V(\rho^{-1}(I))$.
(d) Any finite morphism is proper (see also Exercise 3.22).
3.3.18. Let $K$ be a number field. We know that the ring of integers $\mathcal{O}_K$ of $K$ is finite over $\mathbb{Z}$. Show that if $X$ is separated connected scheme containing $\text{Spec} \, \mathcal{O}_K$ as an open subscheme, then $X = \text{Spec} \, \mathcal{O}_K$ (use Exercises 3.17(d) and 3.5).

3.3.19. Let $X$ be a proper algebraic variety over a field $k$. Let $f : X \to Y$ be a morphism of $k$-schemes with $Y$ affine. Show that $f(X)$ is a finite set of closed points (hint: $f$ factors into $X \to \text{Spec} \, \mathcal{O}_X(x) \to Y$).