2.3.1. Let \( X \) be an open subscheme of an affine scheme \( Y \). Show that the canonical morphism \( X \to Y \) corresponds by the map \( \rho \) of Proposition 3.25 to the restriction \( O_Y(Y) \to O_X(X) \).

2.3.2. Let \( U = \text{Spec} \ B \) be an affine open subscheme of \( X = \text{Spec} \ A \). Show that the restriction \( A \to B \) is a flat homomorphism.

Let \( i : \text{Spec} \ B = U \to X = \text{Spec} \ A \) be the open immersion and write \( f : A \to B \) for the corresponding ring homomorphism. Since \( i \) is an open immersion, for every prime ideal \( q \) of \( B \) the ring homomorphism \( A_{f^{-1}(q)} \to B_q \) induced by \( f \) is an isomorphism. So by Corollary 1.2.15(2), \( f \) is flat.

2.3.3. Closure of a subset.

(a) Let \( F \) be a subset of an affine scheme \( \text{Spec} \ A \). Show that the closure \( \overline{F} \) of \( F \) in (the underlying topological space of) \( \text{Spec} \ A \) is equal to \( V(I) \), where \( I = \bigcap_{p \in F} p \).

(b) Let \( \varphi : A \to B \) be a ring homomorphism. Let \( f : \text{Spec} \ B \to \text{Spec} \ A \) be the morphism of schemes associated to \( \varphi \). Show that \( \text{im} \ f = V(\ker \varphi) \). Study the situation where \( B \) is the localization of \( A \) at a prime ideal.

(a) Let \( C \) be a closed set which contains \( F \). We have \( C = V(J) \) for some ideal \( J \) of \( A \). So every \( p \in F \) contains \( J \), hence \( I \supseteq J \). Thus \( C = V(J) \supseteq V(I) \). So \( V(I) \) is contained in every closed subset of \( \text{Spec} \ A \) that contains \( F \); thus \( V(I) \subseteq \overline{F} \). Conversely, every \( p \in F \) contains \( I \), so \( p \in V(I) \); hence \( F \subseteq V(I) \). As \( V(I) \) is closed, we get \( \overline{F} \subseteq V(I) \).

(b) Using (a), we have

\[
\overline{\text{im} \ f} = V \left( \bigcap_{p \in \text{im} \ f} p \right)
= V \left( \bigcap_{q \in \text{Spec} \ B} f(q) \right)
= V \left( \bigcap_{q \in \text{Spec} \ B} \varphi^{-1}(q) \right)
= V \left( \varphi^{-1} \left( \bigcap_{q \in \text{Spec} \ B} q \right) \right)
= V \left( \varphi^{-1}(\sqrt{0}) \right)
= V \left( \sqrt{\varphi^{-1}(0)} \right)
= V \left( \varphi^{-1}(0) \right) = V(\ker \varphi).
\]
2.3.4. Let $X$ be a scheme and $f \in \mathcal{O}_X(X)$. Show that $U \mapsto f|_U \mathcal{O}_X(U)$ for every affine open subset $U$ defines a sheaf of ideals on $X$. We denote this sheaf by $f\mathcal{O}_X$. Show that $\text{Supp } f\mathcal{O}_X$ is closed (see Exercise 2.5).

For every affine open $U$, evidently $(f\mathcal{O}_X)(U) = f|_U \mathcal{O}_X(U)$ is an ideal of $\mathcal{O}_X(U)$. And if $U \subseteq V$, the restriction $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ sends $f|_U$ to $f|_V$, so $f\mathcal{O}_X$ is a $\mathcal{B}$-sheaf of ideals on $X$, where $\mathcal{B}$ denotes the collection of affine open subsets. This uniquely extends to a sheaf of ideals on $X$. Next we show that $X \setminus \text{Supp } f\mathcal{O}_X$ is open. To that end, let $x \in X \setminus \text{Supp } f\mathcal{O}_X$. So $(f\mathcal{O}_X)_x = 0$, in particular $f_x = 0$, so $x$ has a neighborhood $U$ such that $f|_U = 0$. This implies that $(f\mathcal{O}_X)(U) = 0$, so for every $y \in U$ we have $(f\mathcal{O}_X)_y = 0$; thus $x \in U \subseteq X \setminus \text{Supp } f\mathcal{O}_X$.

2.3.14. We say that a scheme $X$ is quasi-compact as a topological space (Exercise 1.6). Show that a scheme is quasi-compact if and only if it is a finite union of affine schemes.

This follows from the following facts:

- $X$ has an affine open covering, hence a finite affine open covering if $X$ is quasi-compact.
- Affine schemes are quasi-compact.
- A finite union of quasi-compact spaces is quasi-compact.

2.3.19. Let $K$ be a number field. Let $\mathcal{O}_K$ be the ring of integers of $K$ (i.e., the set of elements of $K$ which are integral over $\mathbb{Z}$). Using the finiteness theorem of the class group $\text{cl}(K)$, show that every open subscheme of $\text{Spec } \mathcal{O}_K$ is affine. See also Exercise 4.1.9.

2.3.20. Let $A$ be a ring, $G$ a finite group of automorphisms of $A$, and $A^G$ the subring of elements of $A$ which are invariant under $G$. Let $p : \text{Spec } A \to \text{Spec } A^G$ denote the morphism induced by the inclusion $A^G \to A$.

(a) Show that $G$ acts naturally on $\text{Spec } A$. Show that $p(x_1) = p(x_2)$ if and only if there exists a $\sigma \in G$ such that $\sigma(x_1) = x_2$.
(b) Show that $A$ is integral over $A^G$ (let $a \in A$, and consider the polynomial $P(T) := \prod_{\sigma \in G}(T - \sigma a) \in A^G[T]$). Deduce from this that $p$ is surjective (Exercise 1.8).
(c) Let $a \in A$. Let $P(T)$ be as above and let us write $P(T) = T^d + \sum_{i \leq d-1} b_i T^i, b_i \in A^G$. Show that $p(D(a)) = \bigcup_i D(b_i)$. Show that $p$ is open (in the topological sense).
(d) Show that for any $b \in A^G$, we have $p^{-1}(D(b)) = D(bA)$ and $(A^G)_b = (A_b)^G$. Let $V$ be an open subset of $\text{Spec } A^G$. Show that $G$ acts on the scheme $p^{-1}(V)$ and that $\mathcal{O}_{\text{Spec } A^G}(V) = \mathcal{O}_{\text{Spec } A(p^{-1}(V))^G}$.

(a) If $\sigma \in G$ and $q$ is a prime ideal of $A$, then $\sigma(q)$ is also a prime ideal. This evidently defines an action of $G$ on $\text{Spec } A$.

Let $q_1, q_2$ be prime ideals of $A$. Assume

$$q_1 \cap A^G = p(q_1) = p(q_2) = q_2 \cap A^G.$$ (*)
Suppose that $q_2 \not\subseteq \sigma(q_1)$ for every $\sigma \in G$. Because $\{\sigma(q_1) : \sigma \in G\}$ is a finite set of prime ideals of $A$, by prime avoidance we get

$$q_2 \not\subseteq \bigcup_{\sigma \in G} \sigma(q_1).$$

So $q_2$ has an element $a$ such that for every $\sigma \in G$ we have $a \notin \sigma(q_1)$. Equivalently, by inverting $\sigma$’s for every $\sigma \in G$ we have $\sigma(a) \notin q_1$. But then because $q_1$ is prime, we get that

$$b := \prod_{\sigma \in G} \sigma(a) \notin q_1.$$

Then $b \in A^G$ but also $b \in q_2$ since $b$ is a multiple of $a$. This contradicts $\ast$.

Thus there exists $\sigma \in G$ such that $q_2 \subseteq \sigma(q_1)$. Similarly there exists $\tau \in G$ such that $q_1 \subseteq \tau(q_2)$. Writing $\eta = \sigma \tau \in G$, we get $q_2 \subseteq \eta(q_2)$. So we have a chain of inclusions

$$q_2 \subseteq \eta(q_2) \subseteq \eta^2(q_2) \subseteq \cdots.$$

But $\eta$ has finite order, so the chain collapses and we get $\eta(q_2) = q_2$. Hence $\sigma(q_1) \subseteq q_2$ so $q_2 = \sigma(q_1)$.

Conversely, assume $q_2 = \sigma(q_1)$ for some $\sigma \in G$. Then

$$p(q_2) = \sigma(q_1) \cap A^G = \sigma(q_1) \cap \sigma(A^G) = \sigma(q_1 \cap A^G) = q_1 \cap A^G = p(q_1).$$

(b) Let $a$ and $P(T)$ be as in the hint. $G$ acts naturally on $A[T]$ with fixing $T$ and here $
\{T - \sigma a : \sigma \in G\}$ is a $G$-orbit. Thus $P(T) \in A[T]^G = A^G[T]$. As $P(T)$ is monic and $P(a) = 0$ we conclude that $a$ is integral over $A^G$. Because $a \in A$ was arbitrary, $A$ is integral over $A^G$. Since $A^G \to A$ is an integral and injective ring homomorphism, the induced map $p : \text{Spec} A \to \text{Spec} A^G$ is surjective.

(c) Let $q \in D(a)$, which means $a \notin q$. We want to show that $p(q) = q \cap A^G$ lies in $\bigcup_i D(b_i)$ (from here, the $\subseteq$ part of the desired equality follows). Indeed, since

$$- \sum_{i \leq d - 1} b_ia^i = a^d \notin q$$

($q$ is prime), there exists some $i$ such that $b_i \notin q$, equivalently $b_i \notin q \cap A^G$. Thus, $q \cap A^G \subseteq \bigcup_i D(b_i)$.

For the converse, suppose for the sake of contradiction that $\bigcup_i D(b_i) \not\subseteq p(D(a))$. So the subset

$$\{i : D(b_i) \not\subseteq p(D(a))\}$$

of $\{1, \ldots, d - 1\}$ is nonempty, so it has a least element, say $j$. Pick $q' \in D(b_j)$ such that $q' \notin p(D(a))$. As $p$ is surjective, there exists $q \in \text{Spec} A$ such that $q \cap A^G = p(q) = q'$. Therefore $q \notin D(a)$, that is, $a \in q$. ?????????????????????????????????

2.3.21. Let $X$ be a scheme, and let $G$ be a finite group acting on $X$ (i.e., $G$ is endowed with a group homomorphism $G \to \text{Aut}(X)$). We define the quotient scheme $X/G$ by the universal property of Exercise 2.14, where we replace the ringed topological spaces by schemes. It does not always exist.

(a) Let $A$ be a ring, and $G$ a finite group of automorphisms of $A$, which we identify with a group of automorphisms of the scheme $\text{Spec} A$. Show that the quotient scheme $(\text{Spec} A)/G$ exists and is isomorphic to $\text{Spec}(A^G)$. Show that this is also the quotient as a ringed topological space (Exercise 2.14).
(b) Let $U$ be an open subscheme of $\text{Spec } A$ that is stable under $G$. Preserving the notation of the preceding exercise, show that the quotient scheme $U/G$ exists and is isomorphic to $p(U)$.

(c) Let $G$ be a finite group acting on a scheme $X$. We suppose that every point $x \in X$ has an affine open neighborhood that is stable under $G$ (see Exercise 3.3.23 for examples of such $X$). Show that the quotient scheme $X/G$ exists.

Remark. See [72] for quotients of algebraic varieties by algebraic groups.

2.3.22. Let $k$ be a field of characteristic 0. We let $G := \mathbb{Z}$ act on the polynomial ring $k[T]$ by $n : T \mapsto T + n$ if $n \in \mathbb{Z}$.

(a) Show that $G$ is a subgroup of the group of automorphisms (of $k$-algebras) of $k[T]$. We identify it with a group of automorphisms of $\mathbb{A}^1_k$. Show that the only open subschemes of $\mathbb{A}^1_k$ that are stable under $G$ are $\emptyset$ and $\mathbb{A}^1_k$ itself.

(b) Show that the quotient scheme $\mathbb{A}^1_k/G$ exists and is equal to $\text{Spec } k$.

(c) Show that $\mathbb{A}^1_k/G$ is not the quotient as a ringed topological space (Exercise 2.14).