2.3.5. Let $Y$ be a scheme that satisfies the conclusion of Proposition 3.25 for every affine scheme $X$. Show that $Y$ is affine.

Let $\mathbf{Sch}$ be the category of schemes. We have the global sections functor $G : \mathbf{Sch} \to \mathbf{Ring}$ which is contravariant. We also have the spectrum functor $\text{Spec} : \mathbf{Ring} \to \mathbf{Sch}$ which is adjoint with $G$. We are assuming that there is a natural bijection

$$\text{Hom}_{\mathbf{Sch}}(X, Y) \simeq \text{Hom}_{\mathbf{Ring}}(G(Y), G(X)).$$

for every scheme $X$. In particular, if $X = \text{Spec} A$ is affine, we have

$$\text{Hom}_{\mathbf{Sch}}(X, Y) \simeq \text{Hom}_{\mathbf{Ring}}(G(Y), G(X)) \simeq \text{Hom}_{\mathbf{Ring}}(G(Y), A) \simeq \text{Hom}_{\mathbf{Sch}}(\text{Spec} A, \text{Spec} G(Y)).$$

By the proof of Proposition 3.25, we actually get natural bijections

$$\text{Hom}_{\mathbf{Sch}}(X, Y) \simeq \text{Hom}_{\mathbf{Ring}}(G(Y), G(X)) \simeq \text{Hom}_{\mathbf{Sch}}(X, \text{Spec} G(Y)).$$

for every scheme $X$. This means that there is an isomorphism of contravariant functors

$$\text{Hom}_{\mathbf{Sch}}(-, Y) \cong \text{Hom}_{\mathbf{Sch}}(-, \text{Spec} G(Y)) : \mathbf{Sch} \to \mathbf{Set}.$$

Hence by Yoneda’s lemma $Y \cong \text{Spec} G(Y)$ as a scheme so $Y$ is affine.

2.3.6. Generalize Example 3.32 to the case when $k$ is an arbitrary ring.

2.3.7. Let $X$ be a scheme over a field $k$. Let $\varphi : k[T_1, \ldots, T_n] \to \mathcal{O}_X(X)$ be a homomorphism of $k$-algebras and $f : X \to \mathbb{A}_k^n$ the morphism induced by $\varphi$. Show that for any rational point $x \in X(k)$, via the identification $\mathbb{A}_k^n(k) = k^n$, we have $f(x) = (f_1(x), \ldots, f_n(x))$, where $f_i = \varphi(T_i)$ and $f_i(x)$ is the image of $f_i$ in $k(x) = k$.

2.3.8. Let $X$ be a scheme over a ring $A$. Let $f_0, \ldots, f_n \in \mathcal{O}_X(X)$ be such that $f_{i,x}$ generate the unit ideal of $\mathcal{O}_{X,x}$ for every $x \in X$. Show that $X$ is the union of $X_{f_i}$ (see Definition 3.11), and that we have a morphism $f : X \to \mathbb{P}_A^n$ such that $f^{-1}(D_+(T_i)) = X_{f_i}$ and that $f|_{X_{f_i}}$ is induced by the homomorphism $A[T_1^{-1}T_j]_j \to \mathcal{O}_X(X_{f_i})$ given by $T_i^{-1}T_j \mapsto f_i^{-1}f_j$. If $A = k$ is a field and $x \in X(k)$, determine $f(x)$ as in the preceding exercise.

2.3.9. Let $B$ be a graded ring. Let $f \in B$ be non-nilpotent, homogenous of degree 0. Show that $B_f$ possesses a natural grading, and that $D_+(f) \cong \text{Proj} B_f$. 
2.3.10. Let $A$ be a ring, $X = \mathbb{P}^n_A$. Show that $\mathcal{O}_X(X) = A$ (use the covering $\mathcal{U} = \{D_+(T_i)\}_{i}$ of $X$ and the complex associated to $\mathcal{U}$ and $\mathcal{O}_X$ as in Lemma 2.7). Deduce from this that $X$ is affine if and only if $n = 0$.

2.3.11. Let $B = \bigoplus_{d \geq 0} B_d$ be a graded ring. Let $e \geq 1$ be an integer.

(a) Let us denote the graded ring $\bigoplus_{d \geq 0} B_{de}$ by $C$ (so $C_d := B_{de}$). Show that $\text{Proj} \ B \cong \text{Proj} \ C$.

(b) Let us suppose that for every $d \geq 1$, $B_{ed}$ is generated by $B_{d}^{e}$, and that $B_{e}$ is finitely generated over a ring $A$. Show that $\text{Proj} \ B$ is a projective scheme over $A$.

2.3.12. Let $B$ be a Noetherian graded ring.

(a) Show that for any homogenous $f \in B_+$, $B(f)$ is Noetherian.

(b) Show that $\text{Proj} \ B$ is Noetherian.

(a) Since $B$ is Noetherian, the localization $C := B_f$ is Noetherian. $C$ is a $\mathbb{Z}$-graded ring where

$$C_d = \left\{ \frac{b}{f^n} : b \in B \text{ is homogenous and } \deg b - n \deg f = d \right\}.$$ So if we let $C_- = \bigoplus_{d < 0} C_d$, $C_-$ is a homogenous ideal of $C$ such that the quotient ring $E := C / C_- \cong \bigoplus_{d \geq 0} C_d$ is $\mathbb{N}$-graded. Now $E$ is Noetherian and hence $E / E_+ \cong C_0 = B(f)$ is Noetherian.

(b) It suffices to show that $\text{Proj} \ B$ is locally Noetherian and quasi-compact. Since $\{D_+(f) : f \in B_+\}$ is a base of open sets in $\text{Proj} \ B$ and $\mathcal{O}_{\text{Proj} \ B}(D_+(f)) = B(f)$ is Noetherian by (a), $\text{Proj} \ B$ is locally Noetherian.

To see that $\text{Proj} \ B$ is quasi-compact, let $\mathcal{F}$ be a collection of closed sets in $\text{Proj} \ B$ with the finite intersection property (FIP). We may write

$$\mathcal{F} = \{V_+(I) : I \in \mathcal{S}\}$$

where $\mathcal{S}$ is a set of homogenous ideals of $B$. Suppose that $\bigcap_{C \in \mathcal{F}} C = \emptyset$. Then $V_+(\sum_{I \in \mathcal{S}} I) = \emptyset = V_+(B)$, so by Lemma 3.35(b) we have

$$B_+ \subseteq \sqrt{\sum_{I \in \mathcal{S}} I}.$$ Since $B$ is Noetherian, the ideal $B_+$ is finitely generated. So there exists a finite subset $\mathcal{S}_0$ of $\mathcal{S}$ such that

$$B_+ \subseteq \sqrt{\sum_{I \in \mathcal{S}_0} I}.$$ Hence again by Lemma 3.35(b), we get $\bigcap_{I \in \mathcal{S}_0} V_+(I) = V_+(\sum_{I \in \mathcal{S}_0} I) = \emptyset$. This contradicts FIP of $\mathcal{S}$. Hence $\bigcap_{C \in \mathcal{F}} C \neq \emptyset$. Since $\mathcal{F}$ was an arbitrary collection of closed subsets of $\text{Proj} \ B$ with FIP, we conclude that $\text{Proj} \ B$ is quasi-compact.

2.3.13. Let $B = A[X, Y, Z]$ be a polynomial ring. Let $B_d$ be the sub-$A$-module of $B$ generated by the elements of the form $X^a Y^b Z^c$ with $a + 2b + 3c = d$.

(a) Show that $B_d$ induce a grading on $B$. 

(b) Determine $B_6$ and show that $B_{6d}$ is generated by the elements of $B_6^d$. Deduce from this that $\text{Proj} B$ is isomorphic to a closed subscheme of $\mathbb{P}^6_A$.

2.3.15. Let $X$ be a quasi-compact scheme, $A = \mathcal{O}_X(X)$. Let us consider the morphism $f : X \to \text{Spec} A$ induced by the identity on $A$ (Proposition 3.25). Show that $f(X)$ is dense in $\text{Spec} A$.

We prove the statement by making use of Proposition 2.4.2. Consider the closed immersion $i : X_{\text{red}} \to X$. Via the unit of the adjunction of $\text{Spec}$ with the global sections functor, we have a commutative diagram

```
\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
\text{Spec} \mathcal{O}_{X_{\text{red}}}(X) & \xrightarrow{\text{Spec} i^\#} & \text{Spec} \mathcal{O}_X(X)
\end{array}
```

of schemes. Because $X$ is quasi-compact, by Proposition 4.2(c), the kernel of the surjective ring homomorphism $i^\# : \mathcal{O}_X(X) \to \mathcal{O}_{X_{\text{red}}}(X)$ is the nilradical $N(\mathcal{O}_X(X))$. Thus $\text{Spec} i^\#$ is a homeomorphism as a topological map. $i$ is already a homeomorphism, thus we can reduce to the case where $X$ is reduced.

It suffices to show that every (nonempty) principal open set in $\text{Spec} A$ intersects $f(X)$. For the sake of contradiction, suppose not; so there exists $s \in A \setminus \{0\}$ such that $D(s) \cap f(X) = \emptyset$. Note that on a set level, we have

$$f : X \to \text{Spec} A,
\quad x \mapsto \{ t \in A : t_x \in \mathfrak{m}_x \}.$$ 

Therefore for every $x \in X$, we have $\{ t \in A : t_x \in \mathfrak{m}_x \} \not\subseteq D(s)$, which means $s \in \{ t \in A : t_x \in \mathfrak{m}_x \}$, which means $s_x \in \mathfrak{m}_x$. Let $U = \text{Spec} B$ be any affine open subset of $X$. Then $s|_U \in B$ lies in every prime ideal in $B$. But since $X$ is reduced, $B$ is a reduced ring, so $s|_U = 0$. Since $X$ can be covered by affine open subsets, we get $s = 0$; a contradiction.

Note that we haven’t made use of quasi-compactness after assuming $X$ is reduced. So the claim is true for arbitrary reduced schemes as well.

2.3.16. Let $X$ be a scheme. Show that $X$ is locally Noetherian if and only if any affine open subscheme is Noetherian.

First we show that “locally Noetherian” is really a local condition:

**Lemma 1.** Let $X$ be a locally Noetherian scheme. Then the set of open Noetherian subschemes form a base for the topology of $X$.

**Proof.** Let $x \in X$ and $U$ be a neighborhood of $x$ in $X$. By assumption, $x$ has a Noetherian open neighborhood $V$ in $X$. So by Proposition 3.46 part (a), $U \cap V$ is a Noetherian neighborhood of $x$ which is contained in $U$. This verifies the condition for being a base. \qed

It immediately follows from this lemma that an open subset of a locally Noetherian scheme is also locally Noetherian.
To answer the question, the “if” direction is clear because $X$ can be covered by affine open subschemes by definition; since we are assuming each of those are Noetherian we obtain that $X$ is locally Noetherian.

For the converse, suppose that $X$ is locally Noetherian. Let $Y$ be an affine open subscheme of $X$. Then $Y$ is also locally Noetherian. So $Y$ has an open covering of Noetherian schemes, but since $Y$ is quasi-compact $Y$ has a finite open covering of Noetherian schemes, say $Y = \bigcup_{i=1}^n U_i$ where $U_i$ is open and Noetherian. But then each $U_i$ can be covered by finitely many affine open Noetherian schemes, hence $Y$ is covered by finitely many affine open Noetherian schemes. By definition, then $Y$ is Noetherian.

2.3.17. Let $f : X \to Y$ be a morphism of schemes. We say that $f$ is quasi-compact if the inverse image of any affine open subset is quasi-compact.

(a) Show that every closed immersion is quasi-compact.

(b) Show that an open immersion $f : X \to Y$ is quasi-compact if $Y$ is locally Noetherian.

Let us suppose in what follows that $f$ is quasi-compact.

(c) Let $\mathcal{J} = \ker(f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X)$. Show that the ringed topological subspace $Z := V(\mathcal{J})$ of $Y$ (see Lemma 2.23) is a scheme (reduce to the case when $Y$ is affine and draw inspiration from the proofs of Propositions 3.12 and 3.20).

(d) Let $j : Z \to Y$ be the closed immersion. Show that we have a morphism $g : X \to Z$ such that $f = j \circ g$ and that if $f$ decomposes into a morphism $g' : X \to Z'$ followed by a closed immersion $j' : Z' \to Y$, then $Z$ is a closed subscheme of $Z'$.

(e) Show that $f(X)$ is dense in $Z$ (reduce to the case when $Z$ is affine). We call $Z$ the scheme-theoretic closure of $f(X)$ in $Y$. If $X$ is reduced, show that $Z$ is reduced.

(a) Let $V \cong \text{Spec} A$ be an affine open subset of $Y$. The subset $U = f^{-1}(V)$ is open in $X$ and $f|_U : U \to V$ is a closed immersion. Hence by Proposition 2.3.20, $U \cong \text{Spec} A/I$ for some ideal $I$. In particular $U$ is quasi-compact.

(b) Let $V$ be an affine open subset of $Y$. By Exercise 2.3.16, $V$ is Noetherian. Since $f$ is an open immersion, $f^{-1}(V)$ is isomorphic to an open subscheme of $V$, hence is Noetherian by Proposition 2.3.46(a). So a fortiori, $f^{-1}(V)$ is quasi-compact.

(c), (d), (e) ?????????????????????????

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2.3.18. Let $X$ be an affine algebraic variety over a field $k$. Show that there exists a projective variety $\overline{X}$ over $k$ such that $X$ is isomorphic to a dense open subscheme of $\overline{X}$. See also Exercise 3.3.20(b).