Preliminary

Lemma 1. Let \( f : X \to Y \) be a continuous map of topological spaces. If \( A \subseteq X \) is irreducible, then so is \( f(A) \subseteq Y \).

Proof. Suppose \( f(A) \subseteq C \cup D \) where \( C, D \) are closed subsets of \( Y \). Then
\[
A \subseteq f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),
\]
and since \( A \) is irreducible, either \( A \subseteq f^{-1}(C) \) or \( A \subseteq f^{-1}(D) \). Thus either \( f(A) \subseteq C \) or \( f(A) \subseteq D \). This shows that \( f(A) \) is irreducible. \( \square \)

Solutions

2.4.1. Let \( k \) be a field and \( P \in k[T_1, \ldots, T_n] \). Show that \( \text{Spec}(k[T_1, \ldots, T_n]/(P)) \) is reduced (resp. irreducible; resp. integral) if and only if \( P \) has no square factor (resp. admits only one irreducible factor; resp. is irreducible).

Write \( A = k[T_1, \ldots, T_n] \). We may assume \( (P) \) is a proper ideal since by Proposition 2.4.2, \( \text{Spec}(A/(P)) \) is reduced if and only if \( A/(P) \) is a reduced ring and that holds if and only if \( (P) \) is a radical ideal of \( A \).

Note that \( A \) is a UFD, so we can write \( P = p_1^{e_1} \cdots p_n^{e_n} \) where \( p_j \)'s are mutually non-associate irreducibles in \( A \) and \( e_j \in \mathbb{Z}^+ \). Then the element \( f = p_1 \cdots p_n \) lies in \( \sqrt{(P)} \).

So if \( (P) \) is radical, then \( P \) divides \( f \) and this forces \( e_j = 1 \) for all \( j \). Conversely, if \( P \) has no square factor, then \( f = P \). So if \( g \in \sqrt{(P)} \), then \( P \mid g^n \) for some \( n \); and hence \( p_j \mid g^n \) for all \( j \) and that yields \( p_j \mid g \) because \( p_j \) is irreducible. Thus \( P = f \mid g \), that is, \( g \in (P) \).

By Proposition 2.4.7, \( \text{Spec}(A/(P)) \) is irreducible if and only if \( A/(P) \) has a unique minimal prime ideal. Assume that there is a unique prime ideal \( \mathfrak{p} \) of \( A \) which is minimal among the primes containing \( P \). As \( \mathfrak{p} \) is prime, by reordering we may assume \( p_1 \in \mathfrak{p} \). So \( (p_1) \subseteq \mathfrak{p} \). But \( (p_1) \) is also a prime ideal containing \( P \); hence \( \mathfrak{p} = (p_1) \). If \( n \) were larger than 1, we would have \( (p_1) = \mathfrak{p} \subseteq (p_2) \) so the irreducibility of \( p_1 \) forces \( (p_1) = (p_2) \) contradicting the fact that \( p_1 \) and \( p_2 \) are non-associate. Thus \( n = 1 \), that is, \( P \) admits only one irreducible factor. Conversely, assume \( P = p^n \) where \( p \) is an irreducible. Then every prime ideal containing \( P \) also contains \( p \) and hence \( (p) \) is the unique prime ideal which is minimal among the primes containing \( P \).

Finally, \( \text{Spec}(A/(P)) \) is an integral scheme iff \( A/(P) \) is an integral domain iff \( (P) \) is prime iff \( P \) is irreducible.

2.4.2. Let \( X \) be a scheme and \( x \in X \). Show that the image of the morphism \( \text{Spec} \mathcal{O}_{X,x} \to X \) is the set of points of \( X \) that specialize to \( x \).
Let \( y \) be in the image. Then \( y \) lies in an affine open subset \( U \cong \text{Spec} \, A \) which contains \( x \). So we can write \( x = p \) and \( y = q \) where \( p, q \in \text{Spec} \, A \). So \( \mathcal{O}_{X,x} \) gets identified with the localization \( A_p \) and writing \( \varphi : A \to A_p \) for the localization map, we have \( q \in \text{im}(\text{Spec} \, \varphi) \); hence \( q \subseteq p \). By Lemma 4.11 (b) \( y \) specializes to \( x \). And also every such point lies in the image because \( \text{im}(\text{Spec} \, \varphi) = \{ q \in \text{Spec} \, A : q \subseteq p \} \).

2.4.3. Let \( \mathcal{O}_K \) be a discrete valuation ring with field of fractions \( K \) and uniformizing parameter \( t \) (i.e., a generator of the maximal ideal). Show that \( \text{Spec} \, K[T] \) can be identified with an open subscheme of \( \text{Spec} \, \mathcal{O}_K[T] \). Determine the set of closed points of \( \text{Spec} \, K[T] \) which specialize to the point corresponding to the maximal ideal \( (T, t) \) of \( \mathcal{O}_K[T] \). See also Proposition 10.1.40(c).

2.4.4. We say that a scheme \( X \) is connected if the underlying topological space is connected. For example, an irreducible scheme is connected. Let \( X \) be a scheme having only finitely many irreducible components \( \{ X_i \}_i \). Show that \( X \) is connected if and only if for any pair \( i, j \), there exists indices \( i_0 = i, i_1, \ldots, i_r = j \) such that \( X_{i_l} \cap X_{i_{l+1}} \neq \emptyset \) for every \( l < r \). Also show that \( X \) is integral if and only if \( X \) is connected, and integral at every point \( x \in X \).

We may assume \( X \neq \emptyset \). Say \( \{ X_i : i \in I \} \) are the irreducible components of \( X \). For \( i, j \in I \) declare \( i \sim j \) if and only if \( X_i \cap X_j \neq \emptyset \). This defines a reflexive and symmetric relation on \( I \) so its transitive closure is an equivalence relation for which we also write \( \sim \). So we can restate the first part of the question as

“\( X \) is connected iff \( \sim \) yields a single equivalence class on \( I \).”

For every equivalence class \( E \subseteq I \), write \( X_E := \bigcup_{i \in E} X_i \). As \( E \) is a finite set and each \( X_i \) is closed, \( X_E \) is closed. Also by construction, if \( F \) is another equivalence class under \( \sim \), then \( X_E \cap X_F = \emptyset \). But

\[
\bigcup_{E \in I/\sim} \ X_E = \bigcup_{i \in I} X_i = X,
\]

so each \( X_E \) is also open. Moreover, because of the following standard point-set topology lemma and noting that irreducibility is stronger than being connected, we can conclude that \( X_E \) is connected.

Lemma 2. Let \( \{ X_i : i \in E \} \) be a finite collection of connected subspaces of a topological space \( X \), where \( |E| > 1 \). If for every \( i \in E \) there exists \( j \in E \setminus \{i\} \) such that \( X_i \cap X_j \neq \emptyset \), then \( \bigcup_{i \in E} X_i \) is connected.

Proof. Omitted. \( \square \)

Thus \( \{ X_E : E \in I/\sim \} \) is the set of connected components of \( X \). Therefore \( X \) is connected if and only if \( I/\sim \) is a singleton.

For the second part, the “only if” part is clear. For the “if” part, assume \( X \) is connected and integral at every point \( x \in X \). So \( X \) is reduced at every \( x \in X \), which means by definition that \( X \) is reduced. Also by Proposition 2.4.12(b), every point \( x \) has a unique irreducible component passing through it. This means that the irreducible components of \( X \) are mutually disjoint, that is, \( X_i \cap X_j = \emptyset \) whenever \( i, j \in I \) with \( i \neq j \). So \( \sim \) is just equality and therefore by above \( \{ X_i : i \in I \} \) are the connected components of \( X \). But \( X \) is connected, so \( |I| = 1 \) and \( X \) is irreducible.
2.4.5. Let $X$ be a scheme. Show that every irreducible component of $X$ is contained in a connected component (in the topological sense). Show that if $X$ is locally Noetherian, then the connected components are open. If $X$ is Noetherian, then there are only finitely many connected components.

We may assume $X \neq \emptyset$. Let $X_0$ be an irreducible component of $X$. Pick $x \in X_0$. Then the connected component, say $C$, containing $x$ is the union of all connected subsets of $X$ containing $x$. Since $X_0$ is one of these, $X_0 \subseteq C$.

Suppose $X$ is a Noetherian scheme. Then $X$ has finitely many irreducible components. So Exercise 2.4.4 is applicable, and we have shown in its solution that $X$ has finitely many connected components. Note that this implies that the connected components of $X$ are open.

Suppose $X$ is locally Noetherian. To show that the connected components of $X$ are open, it suffices to show that every $x \in X$ has a connected neighborhood. We know that $x$ has a Noetherian neighborhood $U$, so we may pick the connected component of $U$ containing $x$.

2.4.6. Let $A$ be a ring. We say that $e \in A$ is idempotent if $e \neq 0$ and $e^2 = e$. We will say that $e$ is indecomposable if it cannot be written as a sum of two idempotent elements. Let $X$ be a scheme.

(a) Show the equivalence of the following properties:
   (i) $X$ is connected;
   (ii) $\mathcal{O}_X(X)$ has no other idempotent elements than 1;
   (iii) $\text{Spec} \mathcal{O}_X(X)$ is connected.

(b) Show that any local scheme (i.e. the spectrum of a local ring) is connected.

(c) Let us suppose that the connected components of $X$ are open (e.g., $X$ locally Noetherian). Let $U$ be a connected component of $X$. Show that there exists a unique idempotent element $e$ of $\mathcal{O}_X(X)$ such that $e|_U = 1$ and $e|_{X \setminus U} = 0$. Show that this establishes a canonical bijection from the set of connected components onto the set of indecomposable idempotent elements of $\mathcal{O}_X(X)$, the converse map being given by $e \mapsto V(1 - e)$.

(a) (iii) $\Rightarrow$ (ii): We show the contrapositive. Write $A = \text{Spec} \mathcal{O}_X(X)$. Suppose $A$ has an idempotent $e \neq 1$. Then $1 - e$ is also an idempotent and $Ae$, $A(1 - e)$ are nonzero rings such that $A \cong Ae \times A(1 - e)$ via $a \mapsto (ae, a(1 - e))$. Therefore the nontrivial decomposition

$$\text{Spec } A \cong \text{Spec } Ae \sqcup \text{Spec } A(1 - e)$$

shows that $\text{Spec } A$ is disconnected.

(ii) $\Rightarrow$ (i): We show the contrapositive. Suppose $X$ is disconnected. Then $X$ is the disjoint union of two nonempty open subsets, say $U$ and $V$. Then we have a ring isomorphism

$$\mathcal{O}_X(X) \cong \mathcal{O}_X(U) \times \mathcal{O}_X(V).$$

But $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ are nonzero rings, so $\mathcal{O}_X(X)$ has nontrivial idempotents.

(i) $\Rightarrow$ (iii): BILEMEDIM
(b) Let $A$ be a local ring and $e \in A$ be an idempotent. Since $e^2 = e$, if $e \neq 1$, then $e$ is a non-unit therefore lies in the unique maximal ideal, which is the (Jacobson) radical of $A$. Therefore $1 - e$ is a unit, but $1 - e$ is also an idempotent so $1 - e = 1$ and so $e = 0$, a contradiction. Thus $A$ has no nontrivial idempotents and therefore by (a) Spec $A$ is connected.

(c) Note that connected components are always closed, so $X \setminus U$ is open. Hence
\[ O_X(X) \to O_X(U) \times O_X(X \setminus U) \]
\[ s \mapsto (s|_U, s|_{X \setminus U}) \]
is a ring isomorphism and therefore there is a unique $e \in O_X(X)$ such that $e|_U = 1$ and $e|_{X \setminus U} = 0$. Since $(1, 0)$ is an idempotent in the product ring $O_X(U) \times O_X(X \setminus U)$, so is $e$ in $O_X(X)$.

Moreover, $e$ is indecomposable. Suppose not, so $e = f + g$ where $f, g$ are idempotents in $A := O_X(X)$. Since $e = e^2 = fe + ge$, we may assume that $f, g \in Ae$. But then $f, g$ are nontrivial idempotents in the ring $Ae$ and therefore since $Ae \cong O_X(U) \cong O_U(U)$, by (a) $U$ is not connected; a contradiction.

So we may write $\varepsilon(U)$ for $e$ and get a well-defined map
\[ \varepsilon : \{\text{connected components of } X\} \to \{\text{indecomposable idempotents of } O_X(X)\} . \]

Now let $e$ be an indecomposable idempotent of $O_X(X)$. Recall that any subset $S$ of $O_X(X)$ gives rise to the closed subset
\[ V(S) = \{ x \in X : s_x = 0 \text{ for all } s \in S \} \]
of $X$. We claim that $V(e)$ is a connected component of $X$. First, since $1 = e + (1 - e)$ we have
\[ \emptyset = V(e, 1 - e) = V(e) \cap V(1 - e) . \]

Second, if $x \in X$ ??????????????????????????????

2.4.7. We say that a topological space $X$ verifies the separation axiom $T_0$ if for every pair of points $x \neq y$, there exists an open subset which contains one of the points and not the other one. Show that the underlying topological space of a scheme verifies $T_0$.

Let $x, y$ be distinct points in $X$. Let $U \cong \text{Spec } A$ be an affine neighborhood of $x$. If $y \notin U$ then we are done. If $y \in U$, we can make the identifications $x = p$ and $y = q$ where $p$ and $q$ are distinct prime ideals of $A$. Since $p \neq q$, either $p \nsubseteq q$ or $q \nsubseteq p$. WLOG assume $p \nsubseteq q$; so there exists $f \in p - q$. Then $q$ lies in the principal open set $D(f)$ but $p$ doesn’t.

2.4.8. Let $X$ be a quasi-compact scheme (Exercise 3.14). Show that $X$ contains a closed point. See also Exercise 3.3.26 for a counterexample when $X$ is not quasi-compact.

Let’s define a topological space to be cool if it is nonempty and every nonempty closed subset contains a closed point.

Lemma 3. Let $X$ be a topological space which has a finite open covering of cool subsets. Then $X$ contains a closed point.
Proof. We show by induction on \( n \) that if \( X \) can be written as a union of \( n \) cool and open subsets then \( X \) has a closed point. The basis case is easy since then \( X \) is cool and hence the closed subset \( X \) contains a closed point.

Now assume that the claim is true for \( n - 1 \). Suppose \( X = \bigcup_{j=1}^n U_j \) where each \( U_j \) is cool and open. By induction hypothesis, the subspace \( X_1 = \bigcup_{j=2}^n U_j \) contains a closed point \( x \). Note that this means \( \overline{\{x\}} \cap X_1 = \{x\} \). Consider the closed subset \( \{x\} \cap U_1 \) of \( U_1 \). There are two cases:

- \( \{x\} \cap U_1 = \emptyset \). Then
  \[
  \{x\} = (\{x\} \cap X_1) \cup (\{x\} \cap U_1) = \{x\}
  \]
  hence \( x \) is a closed point of \( X \).
- \( \{x\} \cap U_1 \neq \emptyset \). Then by the coolness of \( U_1 \) its closed subset \( \{x\} \cap U_1 \) contains a closed point \( x_1 \) of \( U_1 \). So we have \( x_1 \in \{x\} \) and \( \{x_1\} \cap U_1 = \{x_1\} \). We want to show that \( x_1 \) is a closed point of \( X \). Observe that
  \[
  \{x_1\} \cap X_1 \subseteq \{x\} \cap X_1 = \{x\}.
  \]
  Suppose \( x \in \{x_1\} \). Then either \( x = x_1 \) or \( x \notin U_1 \). In the former case \( x = x_1 \) is closed in both \( X_1 \) and \( U_1 \), hence in \( X \) and we are done. So assume \( x \notin U_1 \). Then \( X \setminus U_1 \) is a closed subset of \( X \) containing \( x \). This forces \( x_1 \in X \setminus U_1 \); a contradiction. Thus \( x \notin \{x_1\} \) and hence \( \{x_1\} \cap X_1 = \emptyset \). Thus
  \[
  \{x_1\} = ((\{x_1\} \cap X_1) \cup (\{x_1\} \cap U_1)) = \{x_1\}
  \]
  which means \( x_1 \) is a closed point in \( X \).

\( \square \)

Note that affine schemes are cool topological spaces because every nonempty closed subset \( V(I) \) in \( \text{Spec } A \) contains a maximal ideal. Thus quasi-compact schemes, which can be covered by finitely many affine schemes contain a closed point.

**2.4.9.** Let \( X \) be a Noetherian scheme. Show that the set of points \( x \in X \) such that \( \mathcal{O}_{X,x} \) is reduced (resp. is an integral domain) is open.

Let’s call a scheme \( X \) **nice** if \( \{x \in X : \mathcal{O}_{X,x} \text{ is reduced} \} \) is open in \( X \). Suppose \( \{U_i\}_i \) is an open covering of \( X \) such that each \( U_i \) is nice. Then

\[
\{x \in X : \mathcal{O}_{X,x} \text{ is reduced} \} = \bigcup_i \{x \in U_i : \mathcal{O}_{X,x} \text{ is reduced} \} = \bigcup_i \{x \in U_i : \mathcal{O}_{U_i,x} \text{ is reduced} \}
\]

is open, hence \( X \) is nice. Therefore we may assume that \( X = \text{Spec } A \) where \( A \) is a Noetherian ring. Suppose \( A_p \) is reduced for some \( p \in \text{Spec } A \). It suffices to find \( f \in A \setminus p \) such that \( A_f \) is reduced. Indeed localizations of reduced rings are reduced, so for every \( q \in D(f) \) the ring \( A_q \) would be reduced.

Consider the ideal

\[
I = \{a \in A : sa = 0 \text{ for some } s \in A \setminus p\}
\]

of \( A \), which is the kernel of the localization \( A \to A_p \). As \( A_p \) is reduced, \( I \) is a radical ideal of \( A \). And since \( A \) is Noetherian, \( I \) has a finite generating set, say \( \{a_1, \ldots, a_n\} \). So there exists \( s_i \in A \setminus p \) such that \( s_i a_i = 0 \) for each \( i \).
Let \( f = \prod_{i=1}^{n} s_i \in A \setminus p \). We claim that \( A_f \) is reduced. So suppose \((a/1)^n = 0\) in \( A_f \). Then \( f^k a^n = 0 \) for some \( k \in \mathbb{Z}^+ \). So \( a^n \in I \) and hence \( a \in I \) as \( I \) is radical. So
\[
a = \sum_{i=1}^{n} c_i a_i
\]
for some \( c_i \in A \). Therefore \( fa = 0 \), thus \( a/1 = 0 \) in \( A_f \). Since every element of \( A_f \) is of the form \( a/1 \) times a unit, this implies that \( A_f \) is reduced.

For the integral domain part, it suffices to show that if \( A_p \) is a domain then so is \( A_f \) (the argument reducing to this is the same as above). Suppose \( b/1 \cdot c/1 = 0 \) in \( A_f \). Then \( f^k bc = 0 \) for some \( k \in \mathbb{Z}^+ \). Hence \( bc \in I \), but since \( A_p \) is a domain \( I \) is a prime ideal, hence WLOG \( b \in I \). Therefore
\[
b = \sum_{i=1}^{n} d_i a_i
\]
for some \( d_i \in A \). Thus \( fb = 0 \), so \( b/1 = 0 \) in \( A_f \).

**2.4.10.** Let \( f : X \to \text{Spec } A \) be a quasi-compact morphism (Exercise 3.17). Let \( I \) be an ideal of \( A \). Show that \( f(X) \subseteq V(I) \) if and only if \( f_{\text{Spec } A}(I) \subseteq \mathcal{O}_X(X) \) is nilpotent.

Assume \( f(X) \subseteq V(I) \). Let \( a \in I \). We want to show that the section \( s = f_{\text{Spec } A}(a) \in \mathcal{O}_X(X) \) is nilpotent. Fix \( x \in X \) and write \( p = f(x) \). Via the local homomorphism \( A_p \to \mathcal{O}_{X,x} \) which maps \( a/1 \) to \( s_x \), we obtain \( s_x \in m_x \) since \( a \in I \subseteq p \).

So if we pick an affine open subset \( U = \text{Spec } B \) of \( X \) and consider \( t = s|_U \in \mathcal{O}_X(U) = B \), then for every \( q \in \text{Spec } B \) the element \( t/1 \in B_q \) lies in the maximal ideal \( qB_q \), that is, \( t \in q \). Thus \( t \) is nilpotent.

Now being quasi-compact, \( X \) has a finite open covering by affine open subsets, say \( U_1, \ldots, U_n \). By above, \( s|_{U_i} \in \mathcal{O}_X(U_i) \) is nilpotent for every \( i = 1, \ldots, n \). So picking \( m > 0 \) such that \( 0 = (s|_{U_i})^m = (s^m)|_{U_i} \) for every \( i \), we get \( s^m = 0 \) since \( \mathcal{O}_X \) is a sheaf.

Conversely, assume \( f(X) \not\subseteq V(I) \), so there exists \( x \in X \) such that \( p = f(x) \) does not contain \( I \). So pick \( a \in I \setminus p \). As \( a/1 \) is a unit in \( A_p \), writing \( s = f_{\text{Spec } A}(a) \), we get that \( s_x \) is a unit in \( \mathcal{O}_{X,x} \). As \( \mathcal{O}_{X,x} \neq 0 \), \( s_x \) is not nilpotent, therefore \( s \) cannot be nilpotent.

**2.4.11.** Let \( f : X \to Y \) be a morphism of irreducible schemes with respective generic points \( \xi_X, \xi_Y \). We say that \( f \) is dominant if \( f(X) \) is dense in \( Y \). Let us suppose that \( X, Y \) are integral. Show that the following properties are equivalent:

(i) \( f \) is dominant;
(ii) \( f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X \) is injective;
(iii) for every open subset \( V \) of \( Y \) and every nonempty open subset \( U \subseteq f^{-1}(V) \), the map \( \mathcal{O}_Y(V) \to \mathcal{O}_X(U) \) is injective;
(iv) \( f(\xi_X) = \xi_Y \);
(v) \( \xi_Y \in f(X) \).

(v) \( \Rightarrow \) (iv): Since \( \{\xi_X\} = X \) and \( f \) is continuous, we have
\[
\xi_Y \in f(X) = f(\{\xi_X\}) \subseteq \{f(\xi_X)\}.
\]
Therefore \( f(\xi_X) \) specializes to \( \xi_Y \). But since \( \xi_Y \) is a (actually the) generic point of \( Y \), by definition this implies that \( f(\xi_X) = \xi_Y \).

(iv) \( \Rightarrow \) (iii): We may assume that \( V \) is nonempty as the statement is vacuously true for \( V = \emptyset \). Since \( \xi_Y \) is the generic point of \( Y \), we have \( \xi_Y \in V \) and similarly \( \xi_X \in U \). Then since \( f(\xi_X) = \xi_Y \) we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_Y(V) & \xrightarrow{a} & \mathcal{O}_X(U) \\
\downarrow{b} & & \downarrow{d} \\
\mathcal{O}_{Y,\xi_Y} & \xrightarrow{c} & \mathcal{O}_{X,\xi_x}
\end{array}
\]

of rings. By Proposition 2.4.18, \( \mathcal{O}_{Y,\xi_Y} \) is a field (which immediately implies that \( c \) is injective) and \( b \) is injective. So \( d \circ a = c \circ b \) is injective and hence \( a \) is injective.

(iii) \( \Rightarrow \) (ii): If \( f^{-1}(V) \neq \emptyset \), then by assumption \( f^\#: \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)) = f^*\mathcal{O}_X(V) \) is injective for every open \( V \) in \( Y \).

**SIKINTI?????????????????**

Thus \( f^\# \) is injective as a sheaf map.

(ii) \( \Rightarrow \) (i): We show the contrapositive. Suppose that \( f \) is not dominant. So there exists a nonempty open subset \( V \) of \( Y \) such that \( V \cap f(X) = \emptyset \), or equivalently \( f^{-1}(V) = \emptyset \).

(i) \( \Rightarrow \) (v): The proof is almost the same with (v) \( \Rightarrow \) (iv). Since \( f(X) \subseteq \{f(\xi_X)\} \), the assumption forces \( \{f(\xi_X)\} = Y \). So \( f(\xi_X) \) specializes to \( \xi_Y \) and hence \( f(\xi_X) = \xi_Y \).

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**2.4.12.** Let \( B \) be a graded ring. Let \( Y \) be a reduced closed subscheme of \( \text{Proj} \, B \).

Show that there exists a homogenous ideal \( I \) of \( B \) such that \( Y \cong \text{Proj} \, B \).

As a topological subspace \( Y \) can be identified with \( V_+(I) \) for a homogenous ideal \( I \) of \( B \) because that’s what the closed subspaces of \( \text{Proj} \, B \) look like.

**Lemma 4.** With the notation above, \( \sqrt{I} \) is also a homogenous ideal.

**Proof.** Suppose that \( \mathfrak{p} \) is a prime ideal containing \( I \). Then for every \( d \geq 0 \)

\[ I = \bigoplus_{d \geq 0} (I \cap B_d) \subseteq \bigoplus_{d \geq 0} (\mathfrak{p} \cap B_d) = \mathfrak{p}^h. \]

and we know by Lemma 3.35 that \( \mathfrak{p}^h \) is prime. So we have shown that \( \mathfrak{p} \in V(I) \) implies \( \mathfrak{p}^h \in V_+(I) \). Hence

\[ \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} \supseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}^h \supseteq \bigcap_{q \in V_+(I)} q \]

but since \( V_+(I) \subseteq V(I) \) the reverse containment is trivial; hence

\[ \sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{q \in V_+(I)} q \]

is homogenous. \( \square \)

Now we have that \( \sqrt{I} \) is homogenous and moreover \( V_+(I) = V_+(\sqrt{I}) \). Thus we may assume \( I = \sqrt{I} \).

The ring \( B/I \) is reduced since its nilradical is \( \sqrt{I}/I = 0 \). Thus the scheme \( \text{Spec} \, B/I \) is reduced. So in particular for every homogenous \( f \in A := B/I \), \( \mathcal{O}_{\text{Spec} \, A}(D(f)) = A_f \).
is a reduced ring. This implies that the subring $A(f)$ of $A_X$ is also reduced. Since $A(f) = \mathcal{O}_{\text{Proj} A}(D_+(f))$ and $\{D_+(f) : f \text{ homogenous}\}$ forms an affine open covering of $\text{Proj} A$, by Proposition 4.2 part (b) $\text{Proj} A = \text{Proj} B/I$ is a reduced scheme which can be identified as a closed subscheme of $\text{Proj} B$ whose underlying topological space is $V_+(I)$.

But by Proposition 4.2 part (d), there is a unique structure of a reduced closed subscheme on $V_+(I)$. Since $\text{Proj} B/I$ and $Y$ are both such subschemes, we get $Y \simeq \text{Proj} B/I$.

**2.5.1.** Let $X$ be a topological space. Let $\{X_i\}_i$ be a covering of $X$ by closed subsets $X_i$. We assume that it is a locally finite covering, that is to say that every point $x \in X$ admits an open neighborhood $U$ which meets only a finite number of $X_i$. Show that $\dim X = \sup_i \dim X_i$.

First, we observe that we can reduce to the case where the covering is finite. So assume that the claim holds for finite coverings by closed subsets. By assumption, every $x \in X$ has a neighborhood $U_x$ such that $U_x$ intersect only finitely many $X_i$. So $\{X_i \cap U_x\}_i$ is a finite covering of $U_x$ by closed subsets of $U_x$. Therefore by our assumption we have

$$\dim U_x = \sup_i \dim (X_i \cap U_x) \leq \sup_i \dim X_i.$$  

But $\{U_x : x \in X\}$ is an open covering of $X$, so

$$\dim X = \sup_i (\dim U_x : x \in X) \leq \sup_i \dim X_i.$$  

Therefore $X = \sup_i \dim X_i$.

Thus we may assume $\{X_i\}_i = \{X_1, \ldots, X_n\}$ is a finite covering of $X$. It suffices to prove the case $n = 2$, the rest follow by induction (the basis case $n = 1$ is trivial) because $X_1$ and $X_2 \cup \cdots \cup X_n$ is also a covering of $X$ by closed subsets.

So we may write $C$ and $D$ to be closed subsets of $X$ such that $C \cup D = X$ where we want to prove that $\dim X \leq \max\{\dim C, \dim D\}$. If $X_0$ is an irreducible component of $X$, then $X_0 = (C \cap X_0) \cap (D \cap X_0)$ so $X_0 \subseteq C \cap X_0$ or $X_0 \subseteq D \cap X_0$, that is, $X_0 \subseteq C$ or $X_0 \subseteq D$. In either case, we have $\dim X_0 \leq \max\{\dim C, \dim D\}$. By Proposition 5.5(c), we have $\dim X \leq \max\{\dim C, \dim D\}$.

**2.5.2.** Let $X$ be a scheme and $Z$ be a closed subset of $X$. Show that for all $x \in X$, we have $\text{codim}(\{x\}, X) = \dim \mathcal{O}_{X,x}$ and $\text{codim}(Z, X) = \min_{z \in Z} \dim \mathcal{O}_{X,z}$.

We first work out some basic facts about irreducible sets and dimension which are not (or just stated without proof) in Liu’s book. If $Y$ is an irreducible subset of $X$, we define $\text{codim}(Y, X)$ in the same way as in Definition 2.5.7, observing that it is equal to $\text{codim}(\overline{Y}, X)$.

**Lemma 5.** Let $X$ be a topological space. Let $Y$ be an irreducible and let $U$ be an open subset of $X$ such that $Y \cap U \neq \emptyset$.

(a) $Y \cap U$ is irreducible.

(b) If $Z$ is an irreducible closed subset of $Y$ such that $Z \subseteq Y$, then $Z \cap U \subseteq Y \cap U$.

(c) $\text{codim}(Y \cap U, U) \geq \text{codim}(Y, X)$.

(d) If $Y \subseteq U$, then $\text{codim}(Y, U) = \text{codim}(Y, X)$. 

Proof. (a) Suppose \( Y \cap U \subseteq A \cup B \) where \( A, B \) are closed in \( X \). Then
\[
Y \subseteq A \cup B \cup (X \setminus U)
\]
and since \( Y \not\subseteq X \setminus U \) by assumption, by the irreducibility of \( Y \) either \( Y \subseteq A \) or \( Y \subseteq B \). In particular, either \( Y \cap U \subseteq A \) or \( Y \cap U \subseteq B \).

(b) We show the contrapositive: assume \( Z \cap U = Y \cap U \). So
\[
Y = (Y \cap U) \cup (Y \setminus U) = (Z \cap U) \cup (Y \setminus U).
\]
As \( Z \subseteq Y \) by assumption, by the irreducibility of \( Z \) either \( Y \subseteq Z \) or \( Y \subseteq Y \setminus U \). In particular, either \( Y \cap U \subseteq Z \) or \( Y \cap U \subseteq Y \setminus U \).

(c) Note that it makes sense to talk about \( \text{codim}(Y \cap U, U) \) because \( Y \cap U \) is irreducible by (a). Now if
\[
Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n
\]
is a chain of irreducible closed subsets of \( X \) containing \( Y \), then by (a) and (b)
\[
Y \cap U \subseteq Y_0 \cap U \subsetneq Y_1 \cap U \subsetneq \cdots \subsetneq Y_n \cap U
\]
is a chain of irreducible closed subsets of \( U \) containing \( Y \cap U \). The assertion follows from this observation.

(d) “\( \geq \)” follows from (c). To see “\( \leq \)”, let
\[
Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n
\]
be a chain of irreducible closed subsets of \( U \) containing \( Y \). Then if we let \( Z_j \) to be the closure of \( Y_j \) in \( X \) for each \( j \), then \( Y_j = Z_j \cap U \) and \( Z_j \) is irreducible for each \( j \). Hence
\[
Y \subseteq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n
\]
is a chain of irreducible closed subsets of \( X \) containing \( Y \).

\[\square\]

Note that Lemma 5(d) says that the codimension can be calculated locally.

Now we solve the first part of the question. Let \( U \cong \text{Spec} \ A \) to be an affine open neighborhood of \( x \) in \( X \) where \( x \) corresponds to \( p \in \text{Spec} \ A \). Then by Lemma 5(d) and Proposition 2.5.8(b), we have
\[
\text{codim} \{x\}, X) = \text{codim}(\{x\}, U) = \text{codim}(\{p\}, \text{Spec} \ A) = \text{codim}(\{p\}, \text{Spec} \ A) = \text{codim}(V(p), \text{Spec} \ A) = \dim A_p = \dim \mathcal{O}_{X,x}.
\]

For the second part, if \( \{Z_\lambda : \lambda \in \Lambda\} \) are the irreducible components of \( Z \), then by definition
\[
\text{codim}(Z, X) = \min \{\text{codim}(Z_\lambda, X) : \lambda \in \Lambda\}
\]
and also
\[
\min_{z \in Z} \dim \mathcal{O}_{X,z} = \min_{z \in Z} \min_{\lambda \in Z_\lambda} \dim \mathcal{O}_{X,z} = \lambda \in \Lambda\}
\]
so we can reduce to the case where \( Z \) is irreducible. For every \( z \in Z \), as \( \overline{\{z\}} \subseteq Z \), we have
\[
\text{codim}(Z, X) \leq \text{codim}(\overline{\{z\}}, X).
\]
Hence $\text{codim}(Z, X) \leq \min_{z \in Z} \text{codim}(\{z\}, X)$. On the other hand, since $Z$ is irreducible, it has a generic point $\xi \in Z$, so in particular $\{\xi\} = Z$. Thus
\[
\text{codim}(Z, X) = \min_{z \in Z} \text{codim}(\{z\}, X) = \min_{z \in Z} \dim O_{X,z}
\]
where the last equality is by the first part.

### 2.5.3 Show the following properties.

(a) Let $Z$ be a closed subset of a topological space $X$. Show that we have $\text{codim}(Z, X) = 0$ if and only if $Z$ contains an irreducible component of $X$. Give an example (with $X$ non-irreducible) where $\text{codim}(Z, X) = 0$ and $\dim Z < \dim X$.

(b) Let $X = \text{Spec} \ O_K[T]$ where $\mathcal{O}_K$ is a discrete valuation ring, with uniformizing parameter $t \neq 0$. Let $f = tT - 1$. Show that the ideal generated by $f$ is maximal. Let $x \in X$ be the corresponding point. Show that $X$ is irreducible, $\dim O_{X,x} = 1$, and that $\text{codim}\{x\}, X) + \dim\{x\} < \dim X$.

(a) Note that if $Z_0$ is irreducible, $\text{codim}(Z_0, X) = 0$ means that $Z_0$ is maximal among irreducible closed subsets of $X$, that is, $Z_0$ is an irreducible component of $X$.

In general, if $\text{codim}(Z, X) = 0$, then (by the definition of codimension for general closed subsets) there exists an irreducible component $Z_0$ of $Z$ such that $\text{codim}(Z_0, X) = 0$. Then by above $Z_0$ must be an irreducible component of $X$. Conversely, if $Z$ contains an irreducible component $X_0$ of $X$, then $X_0$ is also an irreducible component of $Z$. Therefore $\text{codim}(Z, X) \leq \text{codim}(X_0, X) = 0$, hence $\text{codim}(Z, X) = 0$.

For the desired example, by above it suffices to find a scheme $X$ which has two irreducible components of different dimensions; because we may then choose $Z$ to be the irreducible component with smaller dimension, which ensures $\dim Z < \dim X$ while $\text{codim}(Z, X) = 0$. So pick integral domains $A, B$ such that $\dim A < \dim B$ (for instance $A = k[X]$ and $B = k[X, Y]$ where $k$ is a field). Then let $X = \text{Spec}(A \times B) = \text{Spec} A \sqcup \text{Spec} B$ and $Z = \text{Spec} A$, $Z' = \text{Spec} B$. Here $Z$ and $Z'$ are irreducible because $A$ and $B$ are domains, and they are the irreducible components of $X$ by Proposition 2.4.5(c) such that $\dim Z < \dim Z'$.

(b) Note that the quotient ring $\mathcal{O}_K[T]/(f)$ is isomorphic to the localization $(\mathcal{O}_K)_t$.

**Lemma 6.** If $\mathcal{O}_K$ is a discrete valuation ring with uniformizing parameter $t$, then every nonzero proper ideal of $\mathcal{O}_K$ is generated by $t^n$ for some $n \geq 1$.

**Proof.** By the book’s definition, $\mathcal{O}_K$ is a local PID which is not a domain and $(t)$ is the unique maximal ideal. So in particular $\mathcal{O}_K$ is a UFD such that every irreducible element is associate with $t$. Then if $(g)$ is an arbitrary nonzero proper ideal of $\mathcal{O}_K$, then $g$ is nonzero and non-unit so $g = ut^n$ for some unit $u \in \mathcal{O}_K$ and $n \geq 1$. Thus $(g) = (t^n)$. \[\square\]

Every nonzero element of $(\mathcal{O}_K)_t$ can be written of the form $a/t^n$ where $\gcd(a, t^n) = 1$. By the above lemma, the ideal generated by $a$ in $\mathcal{O}_K$ cannot be a proper ideal, hence $a$ must be a unit. Thus $a/t^n$ is a unit in $(\mathcal{O}_K)_t$.

This shows that $(\mathcal{O}_K)_t$ is a field, therefore $m := (f)$ is maximal in $\mathcal{O}_K[T]$.

As $\mathcal{O}_K[T]$ is an integral domain, $X = \text{Spec} \mathcal{O}_K[T]$ is a fortiori irreducible. Note that
\[
\dim O_{X,x} = \dim (\mathcal{O}_K[T])_m = \text{ht}(m).
\]
Thanks to the chain $0 \subseteq \mathfrak{m}$, $\text{ht}(\mathfrak{m}) \geq 1$. On the other hand, by Krull’s principal ideal theorem (which is applicable as $O_K[T]$ is Noetherian and $(f) = \mathfrak{m}$ is prime), we have $\text{ht}(\mathfrak{m}) \leq 1$. Thus $\dim \mathcal{O}_{X,x} = 1$.

Since $x$ is a closed point, by Exercise 2.5.2 we have $\text{codim}(\{x\}, X) = \text{codim}(\overline{\{x\}}, X) = \dim \mathcal{O}_{X,x} = 1$. And trivially $\dim \{x\} = 0$. However, by Example 2.5.4 and Corollary 2.5.17, $\dim X = \dim \mathcal{O}_K[T] = 1 + 1 = 2$.

2.5.4. Let $A$ be a ring and $p_1, \ldots, p_r$ be prime ideals of $A$. Let $I$ be an ideal of $A$ contained in none of the $p_i$. We want to show that $I$ is not contained in $p_1 \cup \cdots \cup p_r$.

(a) Show that the property is true for $r \leq 2$.

(b) Assume that the property is true for $r - 1$ and that $p_r$ does not contain any $p_i$, $i \leq r - 1$. Let $x \in I \setminus (p_1 \cup \cdots \cup p_{r-1})$. Show that there exists a $y \in (Ip_1 \cdots p_{r-1}) \setminus p_r$.

(c) Show that either $x$ or $x + y$ is not in $p_1 \cup \cdots \cup p_r$.

(a) Suppose $I \subseteq p_1 \cup p_2$. Since $I \not\subseteq p_1$, there exists $x_2 \in I \setminus p_1$ and similarly there exists $x_1 \in I \setminus p_2$. Then necessarily $x_1 \in p_1$ and $x_2 \in p_2$. Consider $x_1 + x_2 \in I$. Since $x_1 \in p_1$ and $(x_1 + x_2) - x_1 = x_2 \notin p_1$, we get $x_1 + x_2 \notin p_1$. But similarly $x_1 + x_2 \notin p_2$, so $x_1 + x_2 \in I \setminus (p_1 \cup p_2)$; a contradiction. Note that we didn’t need to use the fact that $p_1, p_2$ are prime here.

(b) Pick $y_i \in p_i \setminus p_r$ for each $1 \leq i \leq r - 1$ and also $y_0 \in I \setminus p_r$. Then since $p_r$ is prime, $y := \prod_{i=0}^{r-1} y_i \notin p_r$. Yet evidently $y \in Ip_1 \cdots p_{r-1}$.

(c) Note that since $y \in p_i$ and $x \notin p_i$ for every $1 \leq i \leq r - 1$, we have $x + y \notin p_i$ for every $1 \leq i \leq r - 1$. Suppose $x \in p_1 \cup \cdots \cup p_r$. Then by the choice of $x$, we have $x \in p_r$. But $y \notin p_r$, so $x + y \notin p_r$. Thus $x + y \notin p_1 \cup \cdots \cup p_r$.

2.5.5. Let $A$ be a graded ring, and let $p_1, \ldots, p_r$ be homogeneous prime ideals of $A$. Let $I$ be a homogeneous ideal of $A$ contained in none of the $p_i$. We want to show that there exists a homogeneous element of $I$ not contained in $\bigcup p_i$. One can suppose that $p_r$ does not contain any $p_i$, $i < r$.

(a) Show that there exists a homogeneous element $a \in Ip_1 \cdots p_{r-1} \setminus p_r$.

(b) Let $b \in I$ be a homogeneous element such that $b \notin \bigcup_{i \leq r-1} p_i$. Show that $b$ or $a^{\deg b} + b^{\deg a}$ is not in $p_1 \cup \cdots \cup p_r$. Conclude.

(a) For each $1 \leq i \leq r - 1$, since $p_i \not\subseteq p_r$ and $p_i$ is a homogeneous ideal, there exists a homogeneous element $a_i \in p_i \setminus p_r$. Similarly there exists a homogeneous element $a_0 \in I \setminus p_r$. Then $a := \prod_{i=0}^{r-1} a_i$ is a homogeneous element which lies in $Ip_1 \cdots p_{r-1}$ but is not in $p_r$ since $p_r$ is prime.

(b) Fix $i \in \{1, \ldots, r - 1\}$. Note that $a^{\deg b} \in p_i$ because $a \in p_i$ and $b^{\deg a} \notin p_i$ because $b \notin p_i$ and $p_i$ is prime. Then the homogeneous element $c := a^{\deg b} + b^{\deg a}$ lies outside $p_i$.

Suppose $b \in p_1 \cup \cdots \cup p_r$. Then by the choice of $b$, we have $b \in p_r$, hence $b^{\deg a} \in p_r$. But $a \notin p_r$, so $a^{\deg b} \notin p_r$ as $p_r$ is prime. Thus $c \notin p_r$ and so $c \notin p_1 \cup \cdots \cup p_r$.

2.5.8. Let $O_K$ be as in Exercise 5.3, and let $K = \text{Frac}(O_K)$, $k = O_K/tO_K$ be the residue field of $O_K$. Let us set $A = K \times k$ and let $\varphi : O_K \to A$ be the homomorphism induced by $O_K \to K$ and $O_K \to k$. Show that $\text{Spec} \varphi : \text{Spec} A \to \text{Spec} O_K$ is
surjective and that \( \dim \mathcal{O}_K > \dim A \). Also show that \( A \) is a finitely generated \( \mathcal{O}_K \)-algebra (i.e., quotient of a polynomial ring over \( \mathcal{O}_K \)).

We know that the only prime ideals of \( \mathcal{O}_K \) are 0 and \( \mathfrak{m} = (t) \). Since \( \mathcal{O}_K \to K \) is injective, \( \varphi \) is injective, so \( \varphi^{-1}(0) = 0 \). And \( \varphi^{-1}(K \times 0) = \mathfrak{m} \). Thus \( \text{Spec} \varphi \) is surjective, as both primes of \( \mathcal{O}_K \) are realized as inverse images under \( \varphi \). However, \( \dim \mathcal{O}_K = 1 \) whereas being an artinian ring, \( \dim A = 0 \).

Also, \( K \cong \mathcal{O}_K[T]/(tT - 1) \) by Exercise 2.5.3 and \( k \cong \mathcal{O}_K/\mathfrak{m} \) so \( A \) is a product of two finitely generated \( \mathcal{O}_K \)-algebras, so \( A \) itself is a finitely generated \( \mathcal{O}_K \)-algebra.

**Lemma 7.** Let \( X \) be a scheme over a field \( k \). Show that the points of \( X(k) \) are closed in \( X \). If \( X \) is an algebraic variety over \( k \), then \( x \in X \) is closed if and only if \( k(x) \) is algebraic over \( k \).

We prove the initial statement first assuming \( X \cong \text{Spec} A \) is affine. Then \( \mathfrak{p} \in X(k) \) means that structure morphism \( X \to \text{Spec} k \) has a section \( \text{Spec} k \to X \) with image \( \mathfrak{p} \). Translating these morphisms between affine schemes to ring homomorphisms, this means that \( A \) is a \( k \)-algebra such that there is a \( k \)-algebra homomorphism \( \varphi : A \to k \) with \( \varphi^{-1}(0) = \mathfrak{p} \). Then \( \varphi \) is surjective and has kernel \( \mathfrak{p} \), therefore \( \mathfrak{p} \) is a maximal ideal, that is, a closed point in \( \text{Spec} A \).

Now let \( X \) be an arbitrary \( k \)-scheme with \( x \in X(k) \). Let \( y \in \overline{\{x\}} \). Then \( y \) has an affine open neighborhood \( U \) such that \( x \in U \). But by Remark 3.31, \( x \in U(k) \) and by what we have shown above,

\[
\overline{\{x\}} \cap U = \{x\}.
\]

So \( y = x \), and hence \( \overline{\{x\}} = \{x\} \).

We follow a similar strategy for the second statement. Let \( A \) be a finitely generated \( k \)-algebra, and \( x \) be a point in \( X = \text{Spec} A \), given by the prime ideal \( \mathfrak{p} \). Then

\[
k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}
\]

is the field of fractions of the integral domain \( A/\mathfrak{p} \). So if \( x \) is a closed point, then \( \mathfrak{p} \) is maximal and hence \( k(x) = A/\mathfrak{p} \) is a finite algebraic extension of \( k \) by Corollary 2.1.12. Conversely, if \( k(x) \) is an algebraic extension of \( k \), then \( A/\mathfrak{p} \) must be a field by the following lemma; so \( \mathfrak{p} \) is maximal and \( x \) is a closed point.

**Lemma 7.** Let \( k \subseteq K \) be an algebraic extension and let \( B \) be a subring of \( K \) that contains \( k \). Then \( B \) is a field.

**Proof.** Let \( b \in B \setminus \{0\} \). Then \( b \in K \) is algebraic over \( k \) so the subring \( k[b] \) of \( B \) is a field; hence \( b \) has an inverse in \( B \). \( \square \)

Now let \( X \) be an arbitrary algebraic variety over \( k \). Then \( X \) is a \( k \)-scheme that has a finite open covering \( X = \bigcup_{i=1}^{n} X_i \) such that \( X_i \cong \text{Spec} A_i \) where \( A_i \) is a finitely generated \( k \)-algebra. Let \( x \in X \). If \( x \) is closed in \( X \), then pick \( X_i \) such that \( x \in X_i \). Then \( x \) is closed in \( X_i \) and so by above \( k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x = \mathcal{O}_{X_i,x}/\mathfrak{m}_{x_i} \) is algebraic over \( k \). Conversely, if \( k(x) \) is algebraic over \( k \), then for each \( X_i \) that contains \( x \), by above \( x \) is closed in \( X_i \). And if \( x \notin X_i \), then \( x \) lies in the closed set \( X \setminus X_i \) and hence \( \{x\} \subseteq X \setminus X_i \).
Thus, for every $X_i$, $\{x\} \cap X_i$ is either empty or equal to $\{x\}$, hence
\[
\{x\} = \bigcup_{i=1}^{n}(\{x\} \cap X_i) = \{x\},
\]
so $x$ is closed in $X$.

2.5.11. (Schemes of dimension 0)

(a) Let $X$ be a scheme which is a (finite or not) disjoint union of open subschemes $X_i$. Show that $\mathcal{O}_X(X) \cong \prod_i \mathcal{O}_X(X_i)$.

(b) Show that any scheme of finite cardinal and dimension 0 is affine.

(c) Let $X = \text{Spec} A$ be a scheme of finite cardinal and dimension 0. Show that every point $x \in X$ is open. Deduce from this that $A \cong \bigoplus_{p \in \text{Spec} A} A_p$.

(d) Show that the statement (c) is false if we do not suppose that $\text{Spec} A$ of dimension 0.

(a) Since $\mathcal{O}_X$ is a sheaf and
\[
X_i \cap X_j = \begin{cases} X_i, & \text{if } i = j \\ \emptyset, & \text{if } i \neq j, \end{cases}
\]
the diagram
\[
\mathcal{O}_X(X) \longrightarrow \prod_i \mathcal{O}_X(X_i) \longrightarrow \prod_i \mathcal{O}_X(X_i),
\]
where the parallel arrows are both the identity, is an equalizer diagram in the category of rings. Thus $\mathcal{O}_X(X) \cong \prod_i \mathcal{O}_X(X_i)$.

(b) Since $X$ has finite cardinal, it has finitely many irreducible components, say $X_1, \ldots, X_n$. Pick any $x \in X_1$. Since $x \subset X_1$ is a chain of irreducible closed subsets, because $\dim X = 0$ we have $x = X_1$. Hence every point of $X_1$ is a generic point of $X_1$, but then by Proposition 4.12 $X_1$ must be a singleton. Similarly every $X_i$ is a singleton, say $\{x_i\}$ and $x_i \neq x_j$ if $i \neq j$. $X = \{x_1, \ldots, x_n\}$ and every point of $X$ is closed, hence also open as $|X| = n$ is finite. Therefore each $\{x_i\}$ must be affine, say isomorphic to $\text{Spec} A_i$ where $A_i$ is a ring with a unique prime ideal. Then
\[
X \cong \bigcup_{i=1}^{n} \text{Spec} A_i \cong \text{Spec} \left( \prod_{i=1}^{n} A_i \right).
\]

(c) This follows immediately from what we’ve shown above in (b).

(d) Let $\mathcal{O}_K$ be a discrete valuation ring with field of fractions $K$. Then 0 and the unique maximal ideal are the only prime ideals of $\mathcal{O}_K$ with respective localizations $K$ and $\mathcal{O}_K$. However $\mathcal{O}_K$ is not isomorphic to $K \times \mathcal{O}_K$ (the latter ring is not even a domain).

2.5.14. Let $K \subseteq L$ be a finite field extension. Let $x \in L$. Then the multiplication by $x$ is an endomorphism of $L$ as a $K$-vector space. We let $\text{Norm}_{L/K}(x)$ denote the determinant of this endomorphism. We also call it the norm of $x$ over $K$.

(a) Show that $\text{Norm}_{L/K}$ is a multiplicative map from $L$ to $K$.

(b) Let $A \subseteq B$ be rings such that $K = \text{Frac}(A)$, $L = \text{Frac}(B)$, and that $B$ is integral over $A$. Show that for any $b \in B$, $\text{Norm}_{L/K}(b)$ is integral over $A$.

(c) Let us moreover suppose that $A$ is a polynomial ring over a field $k$. Show that $\text{Norm}_{L/K}(B) \subseteq A$. 
(a) Clearly $\text{Norm}_{L/K}$ maps $L$ to $K$. Given $x \in L$, let’s write $m_x$ for the endomorphism of $L$ defined by “multiply by $x$”. Then observe that the map

$$\theta : L \to \text{End}_K(L)$$

$$x \mapsto m_x$$

is a ring homomorphism, in particular multiplicative. And $\text{Norm}_{L/K}$ is, by definition, the composition of $\theta$ with the multiplicative map $\det : \text{End}_K(L) \to K$.

(b) Since $L$ is a field, the map $\theta$ above is an embedding. Therefore the minimal polynomial $f \in K[T]$ of the $K$-linear endomorphism $m_b$ is precisely the minimal polynomial of the element $b$ over $K$. So if we let $g \in K[T]$ to be the characteristic polynomial of $m_b$, then by linear algebra we know that

- $g(0) = \pm \det(m_b) = \text{Norm}_{L/K}(b)$,
- the irreducible factors of $g$ and $f$ are the same.

But $f \in K[T]$ is already irreducible, hence $g = f^r$ for some $r$. Thus if $b_1, \ldots, b_n$ are the roots of $f$ in a splitting field, then $g(0)$ is a product of $b_i$’s. But for each $b_i$ is the image of $b$ under some field automorphism fixing $K$, hence is integral over $B$. Thus $g(0)$, and hence $\text{Norm}_{L/K}(b)$, is integral over $B$.

(c) By (b), $\text{Norm}_{L/K}(B)$ lies in the integral closure of $A$ in $K$. If $A = k[T_1, \ldots, T_n]$ then $A$ is a UFD, hence it is integrally closed. Thus $\text{Norm}_{L/K}(B) \subseteq A$. 