1. For any space $X$ and point $p \in X$, there is a constant map $c_p : \Delta^1 \to X$ sending all points to $p$. Viewing this as a 1-dimensional chain in $X$, show that it is always a boundary.

Let $d_p : \Delta^2 \to X$ be the map sending all points of $\Delta^2$ to $p$. Note that for each $i = 0, 1, 2$

$$d_p \circ d_i^2 = c_p.$$  

Thus

$$\partial d_p = c_p - c_p + c_p = c_p.$$

3. Given two paths $\sigma, \gamma : \Delta^1 \to X$ such that $\sigma(1) = \gamma(0)$, we can form a path composite by

$$(\sigma * \gamma)(t) = \begin{cases} 
\sigma(2t) & \text{if } 0 \leq t \leq 1/2 \\
\gamma(2t - 1) & \text{if } 1/2 \leq t \leq 1.
\end{cases}$$

Show that the chain $\sigma + \gamma - (\sigma * \gamma)$ is always a boundary.

Recall that $\Delta^2 = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq v \leq 1\}$. So the map

$$g : \Delta^2 \to [0, 1] = \Delta^1$$

$$(u, v) \mapsto \frac{u + v}{2}$$

is well-defined (and continuous). Thus the map $\alpha = (\sigma * \gamma) \circ g : \Delta^2 \to X$ is a 2-chain in $X$. And its boundary is

$$\partial \alpha = \alpha \circ d_i^2 - \alpha \circ d_0^2 + \alpha \circ d_2^2 \in C_1(X).$$

Recall that by the discussion in class we have

$$d_0^2(t) = (t, 1),$$
$$d_1^2(t) = (t, t),$$
$$d_2^2(t) = (0, t).$$

Thus for every $t \in [0, 1]$,

$$(\alpha \circ d_0^2)(t) = \alpha(t, 1) = (\sigma * \gamma)\left(\frac{t + 1}{2}\right) = \gamma(t),$$
$$\quad (\alpha \circ d_1^2)(t) = \alpha(t, t) = (\sigma * \gamma)(t),$$
$$\quad (\alpha \circ d_2^2)(t) = \alpha(0, t) = (\sigma * \gamma)(t/2) = \sigma(t).$$

Thus $\partial \alpha = \gamma - (\sigma * \gamma) + \sigma$.

2. Given any path $\sigma : \Delta^1 \to X$, write $\overline{\sigma}$ for the same path in the opposite direction (in coordinates, $\overline{\sigma}(t) = \sigma(1 - t)$). Show that the chain $\sigma + \overline{\sigma}$ is always
a boundary. Explain (briefly) some of the extra complications in trying to do something equivalent for maps $\Delta^2 \to X$.

Consider the map

$$\beta : \Delta^2 \to X$$

$$(u,v) \mapsto \sigma(v-u)$$

which is well-defined by the definition of $\Delta^2$. Now $\beta \in C_2(X)$, and since

$$(\beta \circ d^0_2)(t) = \beta(t,1) = \sigma(1-t) = \sigma(t),$$

$$(\beta \circ d^1_2)(t) = \beta(t,t) = \sigma(0),$$

$$(\beta \circ d^2_2)(t) = \beta(0,t) = \sigma(t),$$

using the notation in Problem 1, we have

$$\partial \beta = \sigma - c_{\sigma(0)} + \sigma.$$ 

We know $c_{\sigma(0)}$ is a boundary by Problem 1, hence so is $\sigma + \sigma$.

For a map $\alpha : \Delta^2 \to X$, there seems to be no natural definition of the opposite of $\alpha$ like there is for paths which will “undo” what $\alpha$ does.

4. Using the previous two exercises, explain why an element in $H_1(X)$ can always be represented by a sum of closed loops $S^1 \to X$ with some integer coefficients. (Warning: the same is not even close to true for $H_n(X)$ when $n > 1$.)

For a 1-chain $\sigma : \Delta^1 \to X$, we write $s(\sigma)$ and $t(\sigma)$ for the source and target of $\sigma$, respectively. More explicitly, identifying $\Delta^1$ with $[0,1]$, $s(\sigma) := \sigma(0)$ and $t(\sigma) := \sigma(1)$.

Let $L$ be the subgroup of $C_1(X)$ generated by closed loops, that is,

$$L := \langle \sigma \in C_1(X) : s(\sigma) = t(\sigma) \rangle.$$ 

We will prove that

$$(\star) \quad L + B_1(X) = Z_1(X),$$

equivalently, in $H_1(X) = Z_1(X)/B_1(X)$, every element can be represented by an element in $L$. Note that since a closed loop $\sigma$ satisfies $\partial(\sigma) = 0$, we have $L \subseteq Z_1(X)$. So it remains to show the $\supseteq$ part of $(\star)$.

An arbitrary element in $z \in Z_1(X)$ has the form

$$z = \sum_{\sigma : \Delta^1 \to X} a_\sigma \sigma$$

where only finitely many $a_\sigma$ are zero and $\partial(z) = 0$. By Problem 2, modulo $B_1(X)$, we can replace any $\sigma$ with $-\sigma$; therefore we may assume that each coefficient $a_\sigma$ is nonnegative, i.e., in $\mathbb{N}$.

Let us write $|z|$ for the sum $\sum_{\sigma : \Delta^1 \to X} a_\sigma$ of $z$’s coefficients. Assuming the coefficients of $z, w \in Z_1(X)$ are in $\mathbb{N}$ (which is fine as we saw above), we have $|z + w| = |z| + |w|$.

We prove $(\star)$ by induction on $|z|$. If $|z| = 0$, then $z = 0$ so we are done. Now assume that every $w \in Z_1(X)$ with $|w| < |z|$ lies in $L + B_1(X)$. Write $S = \{ \sigma : \Delta^1 \to X : a_\sigma \neq 0 \}$
for the “support” of $z$. Pick some $\sigma_1 \in S$ and write $c = t(\sigma_1)$. Because

$$0 = \partial(z) = \sum_{\sigma \in S} a_{\sigma} (t(\sigma) - s(\sigma)),$$

we get

$$\sum_{t(\sigma) = c} a_{\sigma} = \sum_{s(\sigma') = c} a_{\sigma'}.$$

As $\sigma_1$ contributes to the left hand sum, there must exist $\sigma_2 \in S$ such that $s(\sigma_2) = c = t(\sigma_1)$. Similarly, there exists $\sigma_3 \in S$ such that $s(\sigma_3) = t(\sigma_2)$ and so on. But

$$\{s(\sigma) : \sigma \in S\} \cup \{t(\sigma) : \sigma \in S\}$$

is a finite set (since $S$ is finite), so there exist $m < n$ such that $t(\sigma_n) = s(\sigma_m)$. By reindexing, we may assume $m = 1$. Now define recursively (using Problem 3’s notation) that

$$\tau_1 = \sigma_1,$$

$$\tau_i = \tau_{i-1} * \sigma_i \quad \text{if } i > 1.$$

Observe that $s(\tau_n) = s(\sigma_1) = t(\sigma_n) = t(\tau_n)$; thus $\tau_n \in L$. Moreover by a repeated application of Problem 3, we have

$$\tau_n \equiv \sum_{i=1}^{n} \sigma_i \quad (\text{mod } B_1(X)).$$

Now observe that the element

$$w = z - \sum_{i=1}^{n} \sigma_i$$

still has nonnegative coefficients and satisfies $|w| < |z|$ because $\sigma_i$’s are in the support of $z$. Finally, by construction $\sum_{i=1}^{n} \sigma_i \in L + B_1(X)$ and by the induction hypothesis, $w \in L + B_1(X)$. Thus $z \in L + B_1(X)$.

5. Given maps $f : A \to B$ and $g : B \to C$ of abelian groups, show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \coker(f) \to \coker(gf) \to \coker(g) \to 0.$$

We show that this follows from the version of the snake lemma which asserts that a commutative diagram of abelian groups

$$\begin{array}{ccc}
A_0 & \xrightarrow{\varphi} & B_0 & \xrightarrow{\alpha} & C_0 & \xrightarrow{\beta} & 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \xrightarrow{} & A_1 & \xrightarrow{\psi} & B_1 & \xrightarrow{} & C_1
\end{array}$$

with exact rows induces an exact sequence

$$\ker(\varphi) \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \coker(\alpha) \to \coker(\beta) \to \coker(\gamma).$$

Moreover it can be checked directly that the kernel of the first map in the above snake sequence is $\ker(\varphi)$ and the cokernel of the last map is $\coker(\psi)$. 
Back to the question. Observe that there is commutative diagram

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} \operatorname{coker} f \xrightarrow{\zeta} 0 \\
\downarrow{gf} \quad \downarrow{g} \quad \downarrow{\zeta} \\
0 \xrightarrow{} C \xrightarrow{id} C \xrightarrow{} 0
\end{array}
\]

Applying snake lemma here, we get an exact sequence

\[
\ker(gf) \to \ker(g) \to \ker(\zeta) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to \operatorname{coker}(\zeta).
\]

Observe that, since \(\zeta\) goes to the 0 group, \(\ker \zeta = \operatorname{coker} f\) and \(\operatorname{coker} \zeta = 0\). Moreover we know that the kernel of the first map in the snake sequence is \(\ker f\). Thus we get the desired exact sequence.