1. Reduced Eilenberg-Steenrod(?)

Let $p$-$Top$ define be the category of pairs of spaces, that is, objects are $(A, X)$ where $X$ is a topological space and $A \subseteq X$ and morphisms are ... you know what they are. Let us assume that there exists a collection of functors such that

$$\{ \tilde{H}_k : p$-$Top \to \text{Ab} \mid k = -2, -1, 0, \cdots \},$$

where $\tilde{H}_{-2}$ is identically the zero functor (this eases the notation), and

(1) **Connecting homomorphism:** For every $k \geq -1$, there is a natural transformation (usually called just $\partial$ without subscripts) from the functor $\tilde{H}_k$ to $\tilde{H}_{k-1} \circ G$ where $G$ is defined by

$$G : p$-$Top \to p$-$Top$$

$$(X, A) \mapsto (A, \emptyset).$$

Writing simply $\tilde{H}_k(Y)$ for $\tilde{H}_k(Y, \emptyset)$, this states that whenever we have a pair $A \subseteq X$ of topological spaces, there is an abelian group homomorphism $\tilde{H}_k(X, A) \to \tilde{H}_{k-1}(A)$ which is natural.

(2) **Long exact sequence:** Given $(X, A) \in p$-$Top$, writing $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$ for the obvious inclusion and identity maps, the sequence

$$\xymatrix{ \tilde{H}_k(A) \ar[r]^{\tilde{h}_k(i)} & \tilde{H}_k(X) \ar[r]^{\tilde{h}_k(j)} & \tilde{H}_k(X, A) \ar[r]^{\partial} & \tilde{H}_{k-1}(A) \ar[r]^{\tilde{h}_{k-1}(i)} & \tilde{H}_{k-1}(X) }$$
is exact for each \( k \geq -1 \). Observe that applying this for every \( k \), we get a long exact sequence

\[
\cdots \longrightarrow \tilde{H}_k(A) \longrightarrow \tilde{H}_k(X) \longrightarrow H_k(X,A) \longrightarrow \tilde{H}_{k-1}(A) \longrightarrow \tilde{H}_{k-1}(X) \longrightarrow H_{k-1}(X,A) \longrightarrow \tilde{H}_{k-2}(A) \longrightarrow \tilde{H}_{k-2}(X) \longrightarrow H_{k-2}(X,A) \longrightarrow \cdots
\]

(3) **Disjoint unions:** For \( k \geq 1 \), \( \tilde{H}_k \) preserves coproducts. Note that the coproduct in \( p\text{-Top} \) is given by the disjoint union (for both the space and subspace in the pair) and the coproduct in \( Ab \) is given by direct sum.

(4) **Excision:** Let \((X,A) \in p\text{-Top}\). If \( V \subseteq A \) satisfies \( V \subseteq \text{Int}(A) \), then for \( k \geq 1 \), \( \tilde{H}_k \) sends the morphism \((X \setminus V, A \setminus V) \to (X,A) \) in \( p\text{-Top} \) to an isomorphism in \( Ab \).

(5) **Homotopy:** Two morphisms \( f,g : (X,A) \to (Y,B) \) in \( p\text{-Top} \) are called homotopic if there is a morphism

\[
H : (X \times [0,1], A \times [0,1]) \to (Y,B)
\]

again in \( p\text{-Top} \) such that \( H(-,0) = f \) and \( H(-,1) = g \). This is an equivalence relation on \( \text{Hom}_{p\text{-Top}}((X,A), (Y,B)) \) and is usually denoted by \( f \simeq g \). The axiom is that if \( f \simeq g \), then \( \tilde{H}_k(f) = \tilde{H}_k(g) \). That is, \( \tilde{H}_k \)'s are homotopy invariant.

(6) **What happens at zero:** For a nonempty set \( X \), let \( \mathcal{P} \) be the set of its path connected components and pick one \( C \in \mathcal{P} \). Then \( \tilde{H}_0(X) \) is the free abelian group generated by the set \( |\mathcal{P} - \{C\}| \). So for instance path connected spaces have \( \tilde{H}_0 \) equal to zero and the two point discrete set has \( \tilde{H}_0 \) equal to \( \mathbb{Z} \). And for the empty set,

\[
\tilde{H}_k(\emptyset) = \begin{cases} 0 & \text{if } k \neq -1, \\ \mathbb{Z} & \text{if } k = -1. \end{cases}
\]

**NOTE:** (3) and (4) are not true for reduced singular homology when \( k = 0 \). The space with two points is a counterexample for both. So even though it nicely kills of the homology of a point, the reduced theory loses some of the axioms at \( k = 0 \).
2. Reduced Homology of Spheres

Now from these assumptions (which are satisfied by the reduced singular homology functors), we compute \( \tilde{H}_k(S^n) \). For precision, note that

\[
S^m = \{ x \in \mathbb{R}^{m+1} : \|x\| = 1 \} \quad \text{and} \quad D^m = \{ y \in \mathbb{R}^m : \|y\| \leq 1 \}
\]

for each \( m \geq 0 \).

Fix \( n \geq 1 \) and \( k \geq 1 \). Observe that \( S^{n-1} \subseteq D^n \). Since \( D^n \) is contractible, that is, homotopy equivalent to a point, its reduced homology vanishes everywhere and hence by the long exact sequence we get

(A) \( \tilde{H}_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1}) \).

Thus we have reduced the calculation of the reduced homology of the sphere to the reduced homology of the point.

Note that since \( S^{n-1} \subseteq D^n - \{0\} \), the identity map \( D^n \to D^n \) yields a morphism \( \iota : (D^n, S^{n-1}) \to (D^n, D^n - \{0\}) \) in \( \text{p-Top} \).

When I was thinking about this part, I thought that \( \iota \) would be a homotopy equivalence in the sense of (5) above but no: the map \( x \mapsto x/\|x\| \) from \( D^n - \{0\} \) to \( S^{n-1} \) cannot be extended to \( D^n \), so we don’t have a homotopy inverse in \( \text{p-Top} \) for \( \iota \). But we can still say that \( \iota \) must induce an isomorphism on reduced homology: indeed, \( \iota \) induces the vertical arrows in the below commutative diagram

\[
\begin{array}{cccccc}
\tilde{H}_k(S^{n-1}) & \longrightarrow & \tilde{H}_k(D^n) & \longrightarrow & \tilde{H}_k(D^n, S^{n-1}) & \longrightarrow & \tilde{H}_{k-1}(S^{n-1}) & \longrightarrow & \tilde{H}_{k-1}(D^n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{H}_k(D^n - \{0\}) & \longrightarrow & \tilde{H}_k(D^n) & \longrightarrow & \tilde{H}_k(D^n, D^n - \{0\}) & \longrightarrow & \tilde{H}_{k-1}(D^n - \{0\}) & \longrightarrow & \tilde{H}_{k-1}(D^n)
\end{array}
\]

where the rows are exact. Here, the first and fourth vertical arrows are simply \( \tilde{H}_k \) and \( \tilde{H}_{k-1} \) applied to the inclusion \( S^{n-1} \to D^n - \{0\} \), which is a homotopy equivalence, yielding isomorphisms. The second and fifth maps are just the identities so we may conclude by the five-lemma that

(B) \( \tilde{H}_k(D^n, S^{n-1}) \cong \tilde{H}_k(D^n, D^n - \{0\}) \).

Pick a point \( p \) from \( S^n \) (we are considering a higher dimensional sphere here!). Then by stereographic projection, there exists a homeomorphism \( \sigma : D^n \to S^n - \{p\} \), which I won’t bother to define (note that this fails when \( n = 0 \), this is why I start with \( n \geq 1 \)). Now consider the origin \( 0 \in D^n \) and write \( q = \sigma(0) \in S^n - \{p\} \). Thus we can consider \( \sigma \) as an isomorphism

\[
\sigma : (D^n, D^n - \{0\}) \to (S^n - \{p\}, S^n - \{p, q\})
\]

in the category \( \text{p-Top} \). Thus we have

(C) \( \tilde{H}_k(D^n, D^n - \{0\}) \cong \tilde{H}_k(S^n - \{p\}, S^n - \{p, q\}) \).

Finally, by applying excision on \( (S^n, S^n - \{q\}) \) by removing the point \( \{p\} \), we get an isomorphism

(D) \( \tilde{H}_k(S^n - \{p\}, S^n - \{p, q\}) \cong \tilde{H}_k(S^n, S^n - \{q\}) \).

Combining (A), (B), (C), (D), we get

\[
\tilde{H}_{k-1}(S^{n-1}) \cong \tilde{H}_k(S^n, S^n - \{q\}) .
\]
On the other hand, applying the long exact sequence for the pair \((S^n, S^n - \{q\})\), since \(S^n - \{q\}\) is contractible, (it is homeomorphic to \(D^n\)), we obtain
\[
\tilde{H}_k(S^n) \cong \tilde{H}_k(S^n, S^n - \{q\}) .
\]
Thus
\[
\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) .
\]
Therefore reducing the indices one by one, in the case \(n > k\) we conclude that
\[
\tilde{H}_k(S^n) \cong \tilde{H}_0(S^{n-k}) = 0
\]
because \(S^{n-k}\) is path connected. And if \(n \leq k\), again proceeding inductively, we get
\[
\tilde{H}_k(S^n) \cong \tilde{H}_{k-n}(S^0) .
\]
But \(S^0\) is a discrete set with two points. Thus we conclude that
\[
\tilde{H}_k(S^n) \cong \begin{cases} Z & \text{if } k = n , \\ 0 & \text{if } k \neq n . \end{cases}
\]