Math 8212  Homework 1  PJW
Date due: Wednesday February 6, 2013. In class on Friday February 8 we will grade your answers, so it is important to be present on that day, with your homework.

Questions 1.1 - 1.6 on page 6 of Matsumura.

Extra questions (not part of the assignment):
As in the book, $A$ is a commutative ring with a 1.

1. Write out a proof that when $k$ is a field, the Jacobson radical of $k[x_1, \ldots, x_n]$ is 0.
2. (From page 3) Write out a proof that if $1 + Ax$ consists of units then $x \in \text{Rad } A$.
3. Assume that $A$ is an integral domain and let $a \in A$. Show that $a$ is irreducible if and only if $(a)$ is maximal among proper principal ideals.
4. Find a ring $A$ and a maximal ideal $I$ of $A[X]$ so that $I \cap A$ is not a maximal ideal of $A$. 
Commutative rings and modules

\[ A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[X]/(X^2 + 5); \] then setting \( k = \mathbb{Z}/2\mathbb{Z} \) we have

\[ A/2A = \mathbb{Z}[X]/(2, X^2 + 5) = k[X]/(X^2 - 1) = k[X](X - 1)^2. \]

Then \( P = (2, 1 - \sqrt{-5}) \) is a maximal ideal of \( A \) containing 2.

Exercises to §1. Prove the following propositions.

1.1. Let \( A \) be a ring, and \( I \subset \text{nil}(A) \) an ideal made up of nilpotent elements; if \( a \in A \) maps to a unit of \( A/I \) then \( a \) is a unit of \( A \).

1.2. Let \( A_1, \ldots, A_n \) be rings; then the prime ideals of \( A_1 \times \cdots \times A_n \) are of the form

\[ A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n, \]

where \( P_i \) is a prime ideal of \( A_i \).

1.3. Let \( A \) and \( B \) be rings, and \( f: A \to B \) a surjective homomorphism.

(a) Prove that \( f(\text{rad } A) \subset \text{rad } B \), and construct an example where the inclusion is strict.

(b) Prove that if \( A \) is a semilocal ring then \( f(\text{rad } A) = \text{rad } B \).

1.4. Let \( A \) be an integral domain. Then \( A \) is a UFD if and only if every irreducible element is prime and the principal ideals of \( A \) satisfy the ascending chain condition. (Equivalently, every non-empty family of principal ideals has a maximal element.)

1.5. Let \( \{P_i \}_{i \in I} \) be a non-empty family of prime ideals, and suppose that the \( P_i \) are totally ordered by inclusion; then \( \bigcap P_i \) is a prime ideal. Also, if \( I \) is any proper ideal, the set of prime ideals containing \( I \) has a minimal element.

1.6. Let \( A \) be a ring, \( I, P_1, \ldots, P_r \) ideals of \( A \), and suppose that \( P_1, \ldots, P_r \) are prime, and that \( I \) is not contained in any of the \( P_i \); then there exists an element \( x \in I \) not contained in any \( P_i \).

2 Modules

Let \( A \) be a ring and \( M \) an \( A \)-module. Given submodules \( N, N' \) of \( M \), the set \( \{a \in A | aN' \subset N \} \) is an ideal of \( A \), which we write \( N:N' \) or \( (N:N)_A \). Similarly, if \( I \subset A \) is an ideal then \( \{x \in M | Ix \subset N \} \) is a submodule of \( M \), which we write \( N:I \) or \( (N:I)_M \). For \( a \in A \) we define \( N:a \) similarly. The ideal \( 0:M \) is called the annihilator of \( M \), and written \( \text{ann}(M) \). We can consider \( M \) as a module over \( A/\text{ann}(M) \). If \( \text{ann}(M) = 0 \) we say that \( M \) is a faithful \( A \)-module. For \( x \in M \) we write \( \text{ann}(x) = \{a \in A | ax = 0 \} \).

If \( M \) and \( M' \) are \( A \)-modules, the set of \( A \)-linear maps from \( M \) to \( M' \) is written \( \text{Hom}_A(M, M') \). This becomes an \( A \)-module if we define the sum \( f + g \) and the scalar product \( af \) by

\[ (f + g)(x) = f(x) + g(x), \quad (af)(x) = a \cdot f(x); \]

(the fact that \( af \) is \( A \)-linear depends on \( A \) being commutative).