Clearly, \( X_i - \alpha_i \) is the minimal polynomial of \( \alpha_i \) over \( k \) for every \( i \). Therefore, by the proof of Theorem 5.1 (iii), the kernel of the map \( k[\alpha_1, \ldots, \alpha_n] \to k(\alpha_1, \ldots, \alpha_n) = k \) is generated by \( X_i - \alpha_i \), \( i = 1, \ldots, n \).

Since \( f \) is in the kernel, we are done.

Let \( m = \text{ht} \mathfrak{p} \). So there exists a chain of prime ideals

\[
P_0 \subset P_1 \subset \ldots \subset P_m = P \subset \mathfrak{p} \quad (X)
\]

in \( R \). By the first part of the proof of Theorem 5.6, the transcendence degree of the quotient ring should decrease at each step in (X). Thus, we have concluded \( (R/\mathfrak{p})_0 \).

**5.2 Definition.** A ring \( R \) is called semiprimary if \( \text{rad} \ R \) is nilpotent and \( R/\text{rad} \ R \) is semisimple.

(A module is semisimple if it is equal to the sum of its simple submodules. A ring is called semisimple if every module over that ring is semisimple.)

**Theorem (Hopkins-Levitzki):** Let \( R \) be a semiprimary ring and \( M \) a (left) \( R \)-module.

The following are equivalent:

(i) \( M \) is Noetherian.

(ii) \( M \) is Artinian.

**Proof:** \((\text{rad} \ R)^n = 0 \) for some \( n \), so we get a filtration (writing \( J = \text{rad} \ R \))

\[
O = J^n M \leq J^{n-1} M \leq \ldots \leq J M \leq M
\]

Here every successive quotient \( J^{n-1} M/J^n M \) is annihilated by \( J \), so it is locally finite on \( R/J \)-module, hence semisimple. Assuming (i), each quotient is also Noetherian, hence Artinian (because under semisimplicity, Artinian \( \Rightarrow \) Noetherian). Thus \( M \) is Artinian. (ii) \( \Rightarrow \) (i) is similar.
We want to show that if \( A \) is a Noetherian zero-dimensional ring, then \( A \) is Artinian. By the above theorem, it suffices to show that \( A \) is semiprimary. (Then we can apply the theorem to the regular module \( _A A \))

Firstly, since all prime ideals are maximal in \( A \), we have
\[
\text{rad} A = \bigcap \text{intersection of all max. ideals}
= \bigcap \text{intersection of all prime ideals}
= \text{nil} A
\]

Hence, every element in \( \text{rad} A \) is nilpotent. As \( A \) is Noetherian, \( \text{rad} A \) is finitely generated. The following lemma shows that \( \text{rad} A \) must be nilpotent:

**Lemma:** Let \( I \) be a finitely generated ideal in a commutative ring \( A \) such that every \( x \in I \) is nilpotent. Then \( I \) is nilpotent.

**Proof:** Write \( I = (x_1, \ldots, x_n) \). Since \( A \) is commutative, for every \( r \in \mathbb{Z}^+ \) we have
\[
I^r = \left\langle x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : a_1 + a_2 + \cdots + a_n = r \right\rangle \quad \text{(this means \( I^r \) is generated by \( \sum \text{such elements} \))}
\]

By assumption, \( x_i^{l_i} = 0 \) for some \( l_i \) (for each \( i \)). So if we choose \( r > \sum_{i=1}^n (l_i + 1) \), we get that
\[
\sum_{i=1}^n a_i = r \Rightarrow a_i > 1 \text{ for some } i.
\]

Thus every generator of \( I^r \) has an \( x_i \) appearing in it, so \( I^r = 0 \). \( \square \)

Finally, we show that \( A/\text{rad} A \) is semisimple. Since \( A \) is zero dimensional, every prime ideal is a minimal prime. But we know that Noetherian rings have finitely many minimal primes. Thus \( A \) has finitely many primes, hence finitely many maximal ideals, say \( m_1, \ldots, m_k \). By Chinese remainder theorem,
\[
A/\text{rad} A = A/m_1 \cdots A/m_k \cong A/m_1 \times \cdots \times A/m_k
\]
being a finite product of fields, this ring is semisimple.
6.6. Clearly \( x_1, x_2 \in (x) \cap (x_2, y) \), so \((x; x, y) \subseteq (x) \cap (x_2, y)\) relatively in the UFD \( k[x, y]\). We have \( x \mid y \cdot h \) and yet \( x, y \) are relatively prime in the UFD \( k[x, y]\); therefore \( x \mid h \). So writing \( h = x \tilde{h} \), we get
\[
f = x^2 g + y x \tilde{h} \in (x^2, xy).
\]
Thus \((x^2, xy) = (x) \cap (x^2, y)\).

Clearly \( x, y, x^2 \in (x) \cap (x^2, xy, y^2)\), so \((x^2, xy) \subseteq (x) \cap (x^2, xy, y^2)\).

Let \( f \in (x) \cap (x^2, x_2, y, y^2)\). So \( x \mid f \) and \( f = x^2 \cdot p + xy \cdot q + y^2 \cdot r \) for some \( p, q, r \in k[x, y] \). Observe that \( x \mid y \cdot r \). Since \( x \) and \( y \) are relatively prime in the UFD \( k[x, y]\), we get \( x \mid r \). So writing \( r = x \tilde{r} \), we get
\[
f = x^2 \cdot p + xy \cdot q + y^2 \cdot x \tilde{r}.
\]
Thus \((x^2, xy) = (x) \cap (x^2, xy, y^2)\).

6.1. Note that in general if we have \( A\)-modules \( M \) and \( N \), then the annihilator \( \text{ann}(x) \cap \text{ann}(y) \), where \((x, y) \in M \oplus N\), is equal to \( \text{ann}(x) \cap \text{ann}(y) \). (Because \( x, y \neq 0 \implies a x = 0 \) and \( a y = 0 \).)

So for \( A = \mathbb{Z} \), \( M = \mathbb{Z} \) and \( N = \mathbb{Z}/3\mathbb{Z} \); for \((x, y) \in M \oplus N\) we have
\[
\text{ann}(x) \cap \text{ann}(y) = \begin{cases} \mathbb{Z} \cap \text{ann}(y) & \text{if } x = 0 \\
\mathbb{Z} \cap \text{ann}(y) & \text{if } x = 0 \\
\{0\} & \text{if } x \neq 0 \\
\text{ann}(y) & \text{if } x = 0 \end{cases}
\]
\[
= \begin{cases} \mathbb{Z} & \text{if } x = 0 \\
\mathbb{Z}/3\mathbb{Z} & \text{if } x = 0 \end{cases}
\]
\[
= \begin{cases} \mathbb{Z} & \text{if } x = 0 \\
\mathbb{Z} & \text{if } x = 0 \end{cases}
\]
\[
= \begin{cases} \mathbb{Z} & \text{if } x = 0 \\
\mathbb{Z} & \text{if } x = 0 \end{cases}
\]
\[
= \begin{cases} \mathbb{Z} & \text{if } x = 0 \\
\mathbb{Z} & \text{if } x = 0 \end{cases}
\]
Thus \( \text{Ass} \left( \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \right) = \{ 0, 3\mathbb{Z} \} \).

Let \( M = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \). Consider the \( \mathbb{Z} \)-submodules of \( M \)

\[
M_1 = \left\{ (x, x+3\mathbb{Z}) : x \in \mathbb{Z} \right\}
\]

\[
M_2 = \left\{ (x, 0) : x \in \mathbb{Z} \right\}
\]

Note that for any \( x, y \in \mathbb{Z} \),

\[
(x, y+3\mathbb{Z}) = (x, 0) + (0, y+3\mathbb{Z})
\]

\[
= (x-0, 0) + (y, y+3\mathbb{Z}) \in M_2 + M_1
\]

so \( M = M_1 + M_2 \).

But by the computation in 6.1, an element in \( M \) has nontrivial annihilator only when the first coordinate is zero. This yields \( \text{Ass}(M_1) = \{ 0 \} \neq \text{Ass}(M_2) \).

But we saw in 6.1 that \( \text{Ass}(M) = \{ 0, 3\mathbb{Z} \} \). So the answer to the question is "No."

1. Let \( \alpha_1 = \sqrt[3]{5} \), \( \alpha_2 = e^{2\pi i/3} \), \( \alpha_3 = e^{2\pi i/3} + 3 \).

   - The minimal polynomial of \( \alpha_1 \) over \( \mathbb{Q} \) is \( x^3 - 5 \), irreducible by Eisenstein.
   - \( \alpha_2 \) is a root of \( x^2 + x + 1 \). Since \( \mathbb{Q}(\alpha_1) \subseteq \mathbb{R} \) and \( \alpha_2 \notin \mathbb{R} \), we have
\( \alpha_2 \notin \mathbb{Q}(\alpha_1) \). Therefore \( x^2 + x + 1 \) is irreducible in \( \mathbb{Q}(\alpha_1)[X] \), hence is the minimal polynomial of \( \alpha_2 \) over \( \mathbb{Q}(\alpha_1) \).

* \( \alpha_3 \) is a root of \( (x - \alpha_1 - \alpha_2)^2 - 2 \in \mathbb{Q}(\alpha_1, \alpha_2)[X] \). To deduce that this is the minimal polynomial of \( \alpha_3 \) over \( \mathbb{Q}(\alpha_1, \alpha_2) \), it suffices to show that \( \alpha_3 \notin \mathbb{Q}(\alpha_1, \alpha_2) \). First, note that the above two items show that

\[
[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] = [\mathbb{Q}(\alpha_2, \alpha_3) : \mathbb{Q}(\alpha_1)] \cdot [\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 2 \cdot 3 = 6
\]

Second, since \( \mathbb{Q}(\alpha_1, \sqrt{2}) \subseteq \mathbb{R} \), we have \( \alpha_2 \notin \mathbb{Q}(\alpha_1, \sqrt{2}) \) and hence

\[
[\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}(\alpha_1, \sqrt{2})] = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}(\alpha_1, \sqrt{2})] = 2.
\]

since \( \alpha_3 = \alpha_1 + \alpha_2 + \sqrt{2} \).

Now, since \( x^2 - 2 \) is the minimal polynomial of \( \sqrt{2} \) over \( \mathbb{Q} \), we have

\[
[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2.
\]

Thus \( 2 \mid [\mathbb{Q}(\alpha_1, \sqrt{2}) : \mathbb{Q}] \). We also have \( [\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 3 \), so \( 3 \mid [\mathbb{Q}(\alpha_1, \sqrt{2}) : \mathbb{Q}] \). Hence \( 6 \mid [\mathbb{Q}(\alpha_1, \sqrt{2}) : \mathbb{Q}] \) and as the degree of the extension cannot be larger, we get the equality

\[
[\mathbb{Q}(\alpha_1, \sqrt{2}) : \mathbb{Q}] = 6.
\]

Collapsing with

Using (1) and (1), we get

\[
[\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] = 12 > 6 = [\mathbb{Q}(\alpha_1, \alpha_1) : \mathbb{Q}]
\]

So \( \alpha_3 \notin \mathbb{Q}(\alpha_1, \alpha_2) \).

By the proof of Theorem 5.1, the kernel is generated by the polynomials

\[
x^3 - 5, \ y^2 + y + 1, \ (z - x - y)^2 - 2.
\]

(6.3) Note that for any \( \lambda \in \mathbb{Z}^* \), \( \lambda A \subseteq \lambda A \) so we have an \( A \)-module homomorphism \( \Pi : A/A^2 \lambda A \to A/A^2 \lambda A \). Note that

\[
\ker \Pi = \{ a \in A/A^2 \lambda A : \Pi(a) = 0 \} = \{ a + \lambda A : a \in \lambda A \}
\]

\[
= \{ a + \lambda A : a \in \lambda A \}
\]

\[
= \{ \lambda a : a \in A \} = im \Pi \]
Note that \( \ker q = \{ a \in A : x a \in x^n A \} \),
\[
\{ a \in A : x a = x^n b \text{ for some } b \in A \}.
\]
Thus, we have an exact sequence
\[
0 \longrightarrow A / x^{n-1} A \longrightarrow A / x^n A \longrightarrow A / x A \longrightarrow 0. \quad (\star)
\]
The claim of the question is a tautology for \( n = 1 \) and using
the induction hypothesis together with \((\star)\) yields
\[
\text{Ass } (A / x^n A) = \text{Ass } (A / x^{n-1} A) \cup \text{Ass } (A / x A) = \text{Ass } (A / x A).
\]
Since \( A / x^{n-1} A \) injects into \( A / x^n A \), we also have
\[
\text{Ass } (A / x A) = \text{Ass } (A / x^{n-1} A) \subseteq \text{Ass } (A / x^n A).
\]
Thus, we get \( \text{Ass } (A / x^n A) = \text{Ass } (A / x A) \) for every \( n \).

By the Noetherian assumption, \( I \) has a primary decomposition \( I = I_1 \cap \ldots \cap I_r \).
Write \( \text{Ass } (A / I_k) = \{ p_k \} \) for each \( k = 1, \ldots, r \).
So \( p_k = \sqrt{\text{ann}_A (A / I_k)} = \sqrt{I_k} \).
Also \( \text{Ass } (A / I) = \{ p_1, \ldots, p_r \} \).

It suffices to show \( J \subseteq I_k \) for every \( k \). By assumption, we have \( J : A p_k \subseteq I : A p_k \subseteq I_k : A p_k \). So for every \( x \in J, \)
\[
\frac{y}{s} \quad \text{for some } y \in I_k \text{ and } s \in A - p_k.
\]
Thus, \( tsx = ty \in I_k \) for some \( t \in A - p_k \). Since \( tsx \in I_k \) and \( ts \notin p_k = I_k \),
we get \( x \in I_k \) as \( I_k \) is a primary ideal.
Note that for \( x \in M \), we have
\[
\begin{align*}
a \in \operatorname{ann}_A(x) & \iff a \cdot x = 0 \\
& \iff f(a) \cdot x = 0 \\
& \iff f(a) \in \operatorname{ann}_B(x) \\
& \iff a \in f^{-1}(\operatorname{ann}_B(x)).
\end{align*}
\]

Thus \( \operatorname{ann}_A(x) = f^{-1}(\operatorname{ann}_B(x)) \). In other words, the inverse image sends annihilators to annihilators. We also know that the inverse image of a prime is prime; therefore there is a well-defined map
\[
f^*: \text{Ass}_B(M) \rightarrow \text{Ass}_A(M)
\]
\[
\begin{array}{c}
P \\
\mapsto f^{-1}(P)
\end{array}
\]

\((f^* is just the restriction of the continuous map \ Spec B \rightarrow \ Spec A)\)

The question asks us to show that \( f^* \) is surjective...

(?)

5.1 Denote the image of \( X_i \) under the natural map \( R \rightarrow R/P \) by \( \alpha_i \); so we have \( R/P = k[x_1, \ldots, x_n] \). Note that
\[
\text{coht}(R) = \dim (R/P) = \text{tr.deg}_k (R/P) \quad (\star)
\]

(\textbf{Theorem 5.6})

So we first prove the special case that \( \text{coht}(P) = 0 \Rightarrow \text{ht}(P) = n \).

Indeed, \( \text{coht}(P) = 0 \) means \((\text{b} \ (\star))\) that \( x_1, \ldots, x_n \) are algebraic over \( k \).

Let \( P_j \) be the kernel of the map
\[
\phi_j: k[x_1, \ldots, x_j, x_{j+1}, \ldots, x_n] \rightarrow k[x_1, \ldots, x_j][x_{j+1}, \ldots, x_n]
\]
\[
\begin{array}{c}
x_i \\
\mapsto \begin{cases} x_i & \text{if } i \leq j \\
X_i & \text{if } i > j
\end{cases}
\end{array}
\]

Note that \( \phi_j \) is surjective and \( k[x_1, \ldots, x_j]_k \) is a field (\( x_1, \ldots, x_j \) are algebraic)
the codomain of \( \phi_j \) is an integral domain. Thus \( P_j \)'s are prime. Moreover by the construction in \textbf{Theorem 5.1}, each \( P_j \) be generated by polynomials which involve every variable \( x_1, \ldots, x_j \) and none of the rest. Thus
\[
0 \subseteq P_1 \subseteq \ldots \subseteq P_n = P
\]
is a strictly increasing chain and hence \( \text{ht}(P) \leq n \). As \( \dim(R) = n \), we get \( \text{ht}(P) = n \).

General case... (?)
Let $P_1, \ldots, P_r$ be the minimal prime ideals of $A$ (there are finitely many because $A$ is Noetherian). Since $A$ is reduced, we have

$$0 = \text{nil } A = P_1 \cap \cdots \cap P_r \quad (\times)$$

So $(\times)$ is a primary decomposition of $0$. Since $\text{Ass } (A/\mathfrak{p}_i) = \{ \mathfrak{p}_i \}$, by Theorem 6.8 (ii) we have $\text{Ass } (A) = \text{Ass } (A/0) \subseteq \{ P_i, \ldots, P_r \}$.

On the other hand, by Theorem 6.5 (iii), the set of minimal elements of $\text{Ass } (A)$ and $\text{Supp } (A) = V(\text{ann } (A)) = V(0) = \text{Spec } A$ coincide. Therefore $P_1, \ldots, P_r \subseteq \text{Ass } (A)$. Thus we have

$$\text{Ass } (A) = \{ P_1, \ldots, P_r \}. \quad \text{[This is not an equality because we don't know a priori that } (\times) \text{ is an irredundant decomposition.]}$$

If we let $S$ to be the multiplicative set of non-zero-divisors of $A$, Theorem 6.1 (ii) yields $S = A - \bigcup_{i=1}^r P_i$. Note that a prime ideal $P$ intersects satisfies $P \cap S = \emptyset$ if and only if $P \subseteq \bigcup_{i=1}^r P_i$, and this happens if and only if $P \subseteq P_j$ for some $j$ (by Exercise 1.6, a.t.a. prime avoidance).

By the minimality of $P_j$, we get $P = P_j$. Thus, the only prime ideals of the localization $A_S$ are $P_1 A_S, \ldots, P_r A_S$.

But these are also the minimal primes of $A_S$ by the correspondence theorem (Theorem 4.1 (ii)). Hence $A_S$ is a zero dimensional ring with maximal ideals $P_1 A_S, \ldots, P_r A_S$. Also note that $P_1 A_S \cap \cdots \cap P_r A_S = (P_1 \cap \cdots \cap P_r) A_S = 0 A_S = 0$.

Therefore, the Chinese remainder theorem yields an isomorphism $A_S \cong A_S / P_1 A_S \times \cdots \times A_S / P_r A_S$, a direct product of fields.