And since 3 corresponds to \( 11 \) in \( \mathbb{Q}_3 \) and
\[
\begin{array}{c}
\text{10101011} \\
\times \quad 11 \\
\hline
\text{10101011} \\
\text{10101011} \\
\hline
\text{00000000} \\
\end{array}
\]
we have \( \text{10101011} = \frac{1}{3} \).

If we shift \( x = \overline{1001101} \) two units to the left of the
decimal point (appending 0s) and add it to itself, we get
\[
\begin{array}{c}
\overline{1001101} \\
\overline{1001101} \\
\hline
\overline{50000000} \\
\end{array}
\]
Shifting two to the left means multiplying with 4 in \( \mathbb{Q}_3 \).
So our computation shows \( x + 4x = 1 \); thus \( x = \frac{1}{5} \).

We know that a \( p \)-adic integer is a positive (rational) integer if and only if it is of the form \( \overline{0} c_n \ldots c_3 c_2 c_1 c_0 \).

If we let \( c_i = c_i \) with not all of \( c_i \)'s zero.

For each \( i \), we have
\[
\begin{array}{c}
\overline{0} c_n \ldots c_{k+1} c_k c_{k-1} \ldots c_1 c_0 \\
+ \overline{1} a_n \ldots a_k a_{k-1} \ldots a_1 a_0 \\
\hline
\overline{0} 0 \ldots 0 0 0 \ldots 0 0 = 0 \\
\end{array}
\]
So let \( k = \min \{ i : c_i = 1 \} \). And define \( a_i = \begin{cases} 0 & \text{if } i < k \\ 1 & \text{if } i = k \\ 1 - c_i & \text{if } i > k \end{cases} \).

Then we have
\[
\begin{array}{c}
\overline{0} c_n \ldots c_{k+1} c_k c_{k-1} \ldots c_1 c_0 \\
+ \overline{1} a_n \ldots a_k a_{k-1} \ldots a_1 a_0 \\
\hline
\overline{0} 0 \ldots 0 0 0 \ldots 0 0 = 0 \\
\end{array}
\]
Thus negative (rational) integers are of the desired form (as they are additive inverses of positive integers).
Let $A$ be the subset of $\mathbb{Q}_\rho$ consisting of strings $a_m \ldots a_n a_2 a_1 a_0 \ldots$. We want to show that $A = \mathbb{Z}_\rho(\rho)$. Let $B$ be the subset of $A$ where the periodic behaviour starts right away. More precisely, $B$ is the subset of $\mathbb{Q}_\rho$ consisting of strings $a_k \ldots a_1 a_0$. Observe that every element $x \in A$ can be written as $x = \rho^s \gamma + n$ where $s > 0$, $\gamma \in B$ and $n \geq 0$.

(To see the periodic part of $x$ to be $\gamma$, then $x$ can be reconstructed by appending a sufficient number of zeros next to $\gamma$ and shifting by a positive integer).

Since $\mathbb{Z} \subseteq \mathbb{Z}_\rho(\rho)$, it suffices to show that $B \subseteq \mathbb{Z}_\rho(\rho)$ to conclude that $A \subseteq \mathbb{Z}_\rho(\rho)$ by this observation.

Take $a_k \ldots a_1 a_0$ in $B$. We have

\[
\begin{align*}
\overline{a_k \ldots a_1 a_0} &= a_0 + a_1 \rho + a_2 \rho^2 + \ldots + a_k \rho^k \\
&= a_0 \rho^k + a_1 \rho^{k+1} + a_2 \rho^{k+2} + \ldots + a_k \rho^{2k+1} \\
&+ a_0 \rho^{2k+2} + a_1 \rho^{2k+3} + \ldots + a_k \rho^{3k+2} \\
&+ a_0 \rho^{3k+3} + a_1 \rho^{3k+4} + \ldots + a_k \rho^{4k+3} \\
&+ \ldots \\
&= a_0 \left(1 + \rho^{k+1} + \rho^{2k+2} + \rho^{3k+3} + \ldots\right) \\
&+ a_1 \rho \left(1 + \rho^{k+1} + \rho^{2k+2} + \rho^{3k+3} + \ldots\right) \\
&+ \ldots \\
&+ a_k \rho^k \left(1 + \rho^{k+1} + \rho^{2k+2} + \rho^{3k+3} + \ldots\right) \\
&= \left(a_0 + a_1 \rho + \ldots + a_k \rho^k\right) \frac{\left(1 + \rho^{k+1} + \rho^{2k+2} + \ldots\right)}{1 - \rho^{k+1}}
\end{align*}
\]

This geometric series type equality makes sense in $\mathbb{Q}_\rho$. Indeed, direct computation yields $(1 - \rho^{k+1})(1 + \rho^{k+1} + \rho^{2k+2} + \ldots) = 1$.
Conversely, we now show that \( Z(p) \subseteq A \). As addition and multiplication are defined inductively on \( Q_p \), eventually periodic elements add and multiply into eventually periodic elements; so \( A \) is a subring of \( Q_p \) which clearly contains the positive integers. Every element in \( Z(p) \) can be written of the form \( \frac{r}{s} \) where \( r \) is a positive integer and \( s \) is an integer coprime to \( p \). Hence it suffices to show that \( \frac{1}{s} \in A \) when \( p \nmid s \). Write the \( p \)-adic expansion of \( s \):

\[
s = a_0 + a_1 p + \cdots + a_k p^k.
\]

Since \( a_0 \neq 0 \), the \( p \)-adic extension of \( s \), \( \frac{1}{s} \), can be constructed inductively. Write

\[
\frac{1}{s} = c_0 + c_1 p + c_2 p^2 + \cdots
\]

The \( c_i \)'s are found by solving the equations:

\[
a_0 c_0 = 1 \pmod{p},
\]

\[
a_0 c_1 + a_1 c_0 + (\text{carry from previous step}) = 0 \pmod{p},
\]

\[
\vdots
\]

Since the \( p \)-adic extension of \( s \) is finite (i.e. \( a_{k+1} = a_{k+2} = \ldots = 0 \)), after some point these equations will repeat themselves, resulting in periodic \( c_i \)'s. Hence we conclude that \( \frac{1}{s} \in A \).

Thus \( A = Z(p) \), as desired. Note that every \( q \in Q \) can be written as \( q = p^t a \) where \( t \geq 0 \) and \( a \in Z(p) \). Let \( p^t \) be the \( p \)-part of the denominator of \( q \). Since multiplying by \( p^t \) corresponds to shifting \( t \) units to the right of the point, we get the given description of \( Q \) since \( Z(p) = A \).

(4) We need an intrinsic definition of exactness in the category Fun(\( C \), Ab) to be able to answer this question without "cheating". Here is an attempt for defining exactness in other categories than module categories. We need some definitions first.

Defn. Let \( C \) be a category. An object \( Z \) is called a zero object in \( C \) if for every \( A \in \text{Obj}(C) \) both \( \text{Hom}_C(Z,A) \) and \( \text{Hom}_C(A,Z) \) are singletons.
In other words, \( Z \) is a zero object iff \( Z \) is both an initial object and a final object.

(For example, the category of rings has no zero object because \( Z \) is initial but the zero ring \( 0 \) is final.)

By their universal property, (if they exist) all zero objects are isomorphic and so we just write \( 0 \) for any of them.

In a category with a zero object \( 0 \), every pair of objects \( A, B \) can be connected by the morphism \( A \rightarrow 0 \rightarrow B \).

This morphism does not depend on which zero object is used in the middle and is called the "zero morphism" and also denoted by \( 0 \).

With the presence of zero morphisms, we can define kernels and cokernels.

Defn: Let \( A \) be a category with a zero object, and \( \eta : A \rightarrow B \) be a morphism in \( A \). A morphism \( \lambda : K \rightarrow A \) is called a kernel of \( \eta \) if \( \eta \circ \lambda = 0 \) and \( \lambda \) is universal with this property.

That is, if \( \xi : Z \rightarrow A \) also satisfies \( \eta \circ \xi = 0 \), then \( \exists ! \xi \) s.t.

\[
\begin{array}{ccc}
K & \overset{\lambda}{\rightarrow} & A \\
\downarrow & & \downarrow \eta \\
Z & \overset{\xi}{\rightarrow} & A
\end{array}
\]

commutes.

Defn: Let \( A \) be a category with a zero object, and \( \eta : A \rightarrow B \) be a morphism in \( A \). A morphism \( \mu : B \rightarrow C \) is called a cokernel of \( \eta \) if \( \mu \circ \eta = 0 \) and \( \mu \) is universal with this property.

That is, if \( \beta : B \rightarrow V \) also satisfies \( \beta \circ \eta \), then \( \exists ! \beta \) s.t.

\[
\begin{array}{ccc}
A & \overset{\eta}{\rightarrow} & B \\
\downarrow \beta & & \downarrow \mu \\
C & \overset{\beta}{\rightarrow} & V
\end{array}
\]

commutes.
Defn: A category \( \mathcal{A} \) with a zero object and every morphism has a kernel and a cokernel is called an exact category.

(This is not a standard usage of the phrase 'exact category'.)

For example for any ring \( R \), the category \( R\text{-Mod} \) is exact because the zero module is a zero object and given an \( R \)-module homomorphism \( \psi : M \to N \); the inclusion \( \ker \psi \to M \) and the projection \( N \to N/\ker \psi \) serve as kernel and cokernel morphisms of \( \psi \), respectively.

The source of a kernel morphism and the target of a cokernel morphism is unique up to isomorphism by their universal properties.

Finally:

Defn: A sequence \( A \xrightarrow{\psi} B \xrightarrow{\psi} C \) in an exact category is called an exact sequence if \( \psi \circ \psi = 0 \) and \( \ker \psi \circ \ker \psi = 0 \).

It can be checked that this definition agrees with the usual notion of exactness in the exact category \( R\text{-Mod} \).

Proposition: Let \( \mathcal{C} \) be a small category and \( \mathcal{A} \) be an exact category. Then \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) is also an exact category.

Proof: First, we need to show that \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) has a zero object.
Define the zero functor \( 0 : \mathcal{C} \to \mathcal{A} \) by assigning each \( X \in \text{Obj}(\mathcal{C}) \) to the zero object in \( \mathcal{A} \) and every morphism in \( \mathcal{C} \) to the identity.

Now given any functor \( F : \mathcal{C} \to \mathcal{A} \), the diagram \( F(X) \to 0 \)

commutes for every \( X, Y \in \text{Obj}(\mathcal{C}) \), \( x \in \text{Hom}_C(X, Y) \) and so we get a natural tran. \( F \to 0 \).
Clearly there is no other nat. tran. from $F$ to $0$ so $0$ is final in $\text{Fun}(C, A)$. Similarly, it is also initial in $\text{Fun}(C, A)$.

Next, we show that $\text{Fun}(C, A)$ has kernels. Let $\nu: F \to G$ be a morphism in $\text{Fun}(C, A)$.

Consider the following assignments:

So $F, G$ are functors from $C$ to $A$ and $\nu: F \to G$ is a natural transformation. So for every $X \in \text{Obj}(C)$, we have a morphism

$$\nu_x : F(X) \to G(X)$$

in $A$. Since $A$ is exact, this morphism has a kernel

$$\lambda_x : K(X) \to F(X)$$

And if $\phi: X \to Y$ is a morphism in $C$, we have a diagram

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\lambda_x} & F(X) \\
\downarrow{F(\phi)} & & \downarrow{\nu_x} \\
K(Y) & \xrightarrow{\lambda_y} & F(Y)
\end{array}
$$

$$
\begin{array}{ccc}
& & G(X) \\
G(\phi) & \downarrow{G(\phi)} & \\
& & G(Y)
\end{array}
$$

We have

$$\nu_y \circ F(\phi) = \lambda_x = G(\phi) \circ \nu_x \circ \lambda_x$$

$$\lambda_x \text{ is a kernel} \Rightarrow \nu_x \circ \lambda_x = 0$$

Since $\lambda_y$ is a kernel of $\nu_y$ $\exists! \ k(\phi): K(X) \to K(Y)$ which makes

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\lambda_x} & F(X) \\
\downarrow{k(\phi)} & & \downarrow{F(\phi)} \\
K(Y) & \xrightarrow{\lambda_y} & F(Y)
\end{array}
$$

commute. From here, it is straightforward, but long, to check that with these assignments $K: C \to A$ defines a functor and the collection of $\lambda_x$'s define a natural transformation from $K$ to $F$. That is, we get a morphism

$$\lambda : K \to F \in \text{Fun}(C, A)$$

and some further diagram chasing shows that $\lambda$ is...
\lambda \text{ satisfies the universal property of being a kernel of } \nu.

The existence of cokernels is similar.

**Proposition:** Let \( \mathcal{A} \) be an exact category and \( \mathcal{C} \) be a small category. A sequence \( F_1 \xrightarrow{\nu} F_2 \xrightarrow{\mu} F_3 \) in \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) is exact if and only if \( F_1(x) \xrightarrow{\nu} F_2(x) \xrightarrow{\mu} F_3(x) \) is exact in \( \mathcal{A} \) for every \( x \in \text{Obj}(\mathcal{C}) \).

**Proof:** \( M \circ \nu = 0 \iff (M \circ \nu)_x = 0 \) in \( \mathcal{A} \) for every \( x \).

\[ \iff \quad M_x \circ \nu_x = 0 \quad \text{for every } x. \]

And since kernels and cokernels in \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) are defined object-wise we have

\[ (\text{coker } \nu) \circ (\ker \mu) = 0 \iff \left( (\text{coker } \nu) \circ (\ker \mu) \right)_x = 0 \quad \text{for every } x \]

\[ \iff \quad (\text{coker } \nu)_x \circ (\ker \mu)_x = 0 \quad \text{for every } x \]

\[ \iff \quad \text{coker } (\nu_x) \circ \ker (\mu_x) = 0 \quad \text{for every } x. \]

\[ \square \]

3. We show that \( \mathbb{Z}_p = \{ \ldots, a_2a_1a_0 : a_0 \in \mathbb{Z}, p \mid 1 \} \) satisfies the universal property of the inverse limit. For each \( k \), define \( \Pi_k : \mathbb{Z}_p \to \mathbb{Z}/p^k \mathbb{Z} \)

\[ \ldots a_2a_1a_0 \to (a_{k-1}p^{k-1} + \ldots + a_1p + a_0) + p^k \mathbb{Z}. \]

In other words, we define \( \Pi_k \) by truncating the string \( \ldots a_2a_1a_0 \) to \( a_{k-1}a_{k-2}a_{k-3} \ldots a_0 \). By the definition of addition and multiplication in \( \mathbb{Z}_p \), \( \Pi_k \) is a ring homomorphism (essentially, \( \Pi_k \) ignores what happens after the \( k \)-th digit and that's nothing but arithmetic modulo \( p^k \)).

Also, if we write \( \varphi_k : \mathbb{Z}/p^k \mathbb{Z} \to \mathbb{Z}/p^{k+1} \mathbb{Z} \), we have

\[ a + p^k \mathbb{Z} \mapsto a + p^{k+1} \mathbb{Z} \]

\[ \left( \varphi_k \circ \Pi_k \right)(\ldots + a_kp^{k} + \ldots + a_1p + a_0) = \varphi_k((a_{k-1}p^{k-1} + \ldots + a_1p + a_0) + p^k \mathbb{Z}) \]

\[ = (a_{k-1}p^{k-1} + \ldots + a_1p + a_0) + p^k \mathbb{Z}. \]
That is, the diagram
\[
\begin{array}{ccc}
\mathbb{Z}_p & \xrightarrow{\Psi_k} & Z/pk\mathbb{Z} \\
\downarrow{\Psi_k} & & \downarrow{q_k} \\
Z/p^k\mathbb{Z} & \xrightarrow{q_k} & Z/p^{k-1}\mathbb{Z}
\end{array}
\]
commutes for each \(k\).

Suppose \(A\) is another ring equipped with ring homomorphisms
\[
\Psi_k : A \rightarrow Z/pk\mathbb{Z}
\]
such that
\[
\begin{array}{ccc}
A & \xrightarrow{\Psi_k} & Z/pk\mathbb{Z} \\
\downarrow{\Psi_k} & & \downarrow{q_k} \\
Z/p^k\mathbb{Z} & \xrightarrow{q_k} & Z/p^{k-1}\mathbb{Z}
\end{array}
\]
commutes for every \(k\).

For each \(x \in A\), the coset \(\Psi_k(x)\) can be represented by a number between 0 and \(p^{k-1}\). So it has a \(p\)-adic expansion of the form
\[
a_{k-1}p^{k-1} + \ldots + a_1p + a_0.
\]
Since the above diagram commutes, the \(p\)-adic expansions for different \(k\)'s will be consistent. For example, \(\Psi_{k+1}(x)\) will have a \(p\)-adic expansion
\[
a_{k+1}p^{k+1} + a_{k-1}p^{k-1} + \ldots + a_1p + a_0
\]
some coefficients that we get from \(\Psi_k(x)\).

Thus we get a well-defined map
\[
\Psi : A \rightarrow \mathbb{Z}_p
\]
where the first \(k\) digits of \(\Psi(x)\) is given by the \(p\)-adic expansion of \(\Psi_k(x)\). Thus the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\Psi} & \mathbb{Z}_p \\
\downarrow{\Psi_k} & & \downarrow{\Psi_k} \\
Z/p^k\mathbb{Z} & \xrightarrow{q_k} & Z/p^{k-1}\mathbb{Z}
\end{array}
\]
commutes for every \(k\) and clearly \(\Psi\) is the unique such map which makes this diagram commute (because it is enough to look at \(\Psi_k(x)\) to recover the \(k\)th digit of \(\Psi(x)\)). It remains to show that \(\Psi\) is a ring homomorphism.
Given \( x, y \in A \), the \( k^{th} \) digit of \( \Psi(x, y) \) is determined by the \( p \)-adic expansion of \( \Psi_k(x, y) = \Psi_k(x) \Psi_k(y) \).

\[
\Psi_k : A \to \mathbb{Z}_p, \quad \text{is a ring homomorphism by assumption}
\]

which is also the \( k^{th} \) digit of the \( p \)-adic expansion of \( \Psi(x) \Psi(y) \). Thus we conclude that \( \Psi(x, y) = \Psi(x) \Psi(y) \). Similarly \( \Psi \) preserves addition and clearly \( \Psi(1) = 1 \).

\( \therefore \Psi \) is a ring homomorphism.

(5) Whenever \( \lambda \leq \mu \), we have \( M_\lambda \supseteq M_\mu \) and we denote the natural map \( M / M_\lambda \to M / M_\mu \) by \( \Psi_{\lambda \mu} \).

We have \( \widehat{M} = \{ (x_\lambda)_{\lambda \in \Lambda} : x_\lambda \in M / M_\lambda \text{ and } \Psi_{\lambda \mu}(x_\lambda) = x_\lambda \text{ whenever } \lambda \leq \mu \} \).

To show that two topologies are the same, we show that every basis element in one topology contains a basis element from the other topology, such that \( x \in B_2 \subseteq B_1 \).

Since singletons form a basis for the discrete topology and product of open sets form a basis for the product topology, a basis element of the "subspace of the product topology" on \( \widehat{M} \) is of the form

\[
B = \left( \prod \{ U_{\lambda} \} \right) \cap \widehat{M}
\]

where \( \Lambda \) has a finite subset \( \Lambda_f \) such that \( U_{\lambda} = \begin{cases} \{ x_\lambda \} & \text{if } \lambda \in \Lambda_f \\ M / M_\lambda & \text{o.w.} \end{cases} \)

So if \( x = (x_\lambda) \in B \) is inside this basis element, we have

\[
U_x = \begin{cases} \{ x_\lambda \} & \text{if } \lambda \in \Lambda_f \\ M / M_\lambda & \text{o.w.} \end{cases}
\]

Since \( \Lambda_f \) is a finite subset of the directed set \( \Lambda \), there exists \( \mu \in \Lambda \).
such that $\lambda \leq \mu$ for each $\lambda \in \Lambda_f$. Now $x + M^*_\mu$ is a basic neighbourhood of $x$ in the linear topology and we claim that $x + M^*_\mu \subseteq B$.

Indeed, let $y \in x + M^*_\mu$. So $y - x \in M^*_\mu$ which means by definition of $M^*_\mu$ that $0 = (y - x)_\mu = y_\mu - x_\mu \sim x_\mu = y_\mu$.

So for every $\lambda \in \Lambda_f$, $\lambda \leq \mu$ and $x_\lambda = \psi_{x,\lambda}(x_\mu) = \psi_{\lambda,\mu}(y_\mu) = y_\lambda$.

Thus $y_\lambda \in U_\lambda$ for every $\lambda$, hence $y \in B = (\prod U_\lambda) \cap \hat{M}$.

Conversely, take a basis element $x + M^*_\mu$ in the linear topology with $y \in x + M^*_\mu$. We want to find a

Let $U_\lambda = \{ \{ x_\lambda \} \text{ if } \lambda = \mu \text{ Then } U = \prod_{\lambda \in \Lambda} U_\lambda, \}

U \cap \hat{M}$ is a basis element in the subspace topology which contains $y$ because $y - x \in M^*_\mu$, i.e. $y_\mu = x_\mu$.

We claim that $y \in U \cap \hat{M} \subseteq x + M^*_\mu$.

shown above

if $z \in U \cap \hat{M}$, $z_\mu = x_\mu$ so $z - x \in M^*_\mu$; i.e. $z \in x + M^*_\mu$.

This finishes the proof.

6. The collection of $M^*_\mu$ is a directed system satisfying the first criterion in section 8. That is, if $x \leq \mu$ then $M^*_x \geq M^*_\mu$.

Indeed, consider the commutative diagram

\[ \begin{array}{cc}
\Lambda & \xrightarrow{P} & \hat{M} \\
M_{x} & \xrightarrow{\sim} & \hat{M}_{x} \\
\end{array} \]
If $\lambda \in \mathcal{M}$, we have $M^*_\lambda \cong M_\lambda$. Therefore, we have a commutative diagram

$$
\begin{array}{ccc}
\hat{M} & \xrightarrow{p_\lambda} & M/ M_\lambda \\
\pi_\lambda \downarrow & & \downarrow q_\lambda \\
M/ M_\lambda & \xrightarrow{\psi_{\lambda \mu}} & M/ M_\mu \\
\end{array}
$$

Since $M^*_\lambda$ is defined to be $\ker(p_\lambda)$, by the commutativity of the diagram we have

$$M^*_\lambda = \ker(p_\lambda) \subseteq \ker(p_\mu) = M^*_\mu.$$

So there is a natural map $\psi_{\lambda \mu} : \hat{M}/M^*_\mu \to \hat{M}/M^*_\lambda$.

Also, observe that the universal property of $\hat{M}$ yields a map $M \to \hat{M}$ which sits in a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\pi_\lambda} & \hat{M} \\
\downarrow & & \downarrow p_\lambda \\
M/ M_\lambda & \to & M/ M_\lambda \\
\end{array}
$$

for each $\lambda$, where $\pi_\lambda$ is the natural projection. Since $\pi_\lambda$ is surjective, $p_\lambda$ is surjective. Since $\ker(p_\lambda) = M^*_\lambda$, we have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & M^*_\lambda & \to & \hat{M} & \xrightarrow{p_\lambda} & M/ M_\lambda & \to & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \to & M^*_\lambda & \to & \hat{M} & \xrightarrow{q_\lambda} & \hat{M}/ M^*_\lambda & \to & 0 \\
\end{array}
$$

with exact rows, and here $f_\lambda$ is an isomorphism. (First isomorphism theorem, if you will.)

Here is the general framework. Since $\lambda \in \mathcal{M} \Rightarrow M^*_\lambda \cong M_\lambda$, regarding the poset $\Lambda$ as a category, we have a contravariant functor

$$F : \Lambda \to \mathbf{A-Mod},$$

$$\lambda \mapsto M/ M_\lambda,$$

$$\lambda \leq \mu \mapsto \psi_{\lambda \mu} : M/ M_\mu \to M/ M_\lambda.$$
And since $\lambda \leq \mu \Rightarrow M^* \cong M_\lambda^*$, we have another contravariant functor:

$$G: \lambda \rightarrow A-\text{Mod}$$

\[ \begin{array}{c}
\chi \leq \mu \\
\psi_{\lambda \mu}: \widehat{M} / M^*_\chi \\
\Rightarrow \widehat{M} / M^*_\mu
\end{array} \]

By definition, we have $\widehat{M} = \varinjlim F$. We will show that $F$ and $G$ are naturally isomorphic functors.

By above, we already have isomorphisms

$$f_\lambda: M / M_\lambda \rightarrow \widehat{M} / M^*_\lambda$$

for each $\lambda$. We need to show naturality, i.e. the assertion that whenever $\chi \leq \mu$, the diagram

\[ \begin{array}{c}
\chi \leq \mu \\
M / M_\chi \\
\downarrow f_\chi \\
\widehat{M} / M^*_\chi
\end{array} \]

\[ \begin{array}{c}
\mu \leq \lambda \\
M / M_\mu \\
\downarrow f_\mu \\
\widehat{M} / M^*_\mu
\end{array} \]

commutes. This can be checked as follows: Insert $\widehat{M}$ on the top:

\[ \begin{array}{c}
\lambda \leq \mu \\
\psi_{\lambda \mu}: \widehat{M} / M^*_\lambda \\
\Rightarrow \widehat{M} / M^*_\mu
\end{array} \]

Now $\psi_{\lambda \mu} \circ f_\mu \circ f_\lambda = \psi_{\lambda \mu} \circ g_\lambda = q_\lambda = f_\lambda \circ \psi_\lambda \circ f_\mu$.

Since $f_\mu$ is surjective, we get

$$\psi_{\lambda \mu} \circ f_\mu = f_\lambda \circ \psi_\lambda \circ f_\mu$$

as desired.
Thus the collection of $f_\lambda$'s define a natural isomorphism between $F$ and $G$. Thus $\hat{M} = \varprojlim F$ and $\varprojlim G$ are isomorphic, via $f_\lambda$'s.

That is, since $\varprojlim G$ constructs precisely the completion of $\hat{M}$ with respect to the family of submodules $\{M_\lambda^*\}_{\lambda \in \Lambda}$, we get that $\hat{M}$ is complete.

Recall that $A$ being I-adically complete means that for every sequence $(a_1, a_2, \ldots)$ which satisfies $a_n - a_{n+1} \in I^n$ for every $n$, there exists a unique element $a \in A$ such that $a - a_n \in I^n$ for every $n$.

Note that by the third paragraph from the bottom of page 55, I-adic and I'-adic topology on $A$ are the same because the two families of ideals $\{I, I^2, I^3, \ldots\}$ and $\{I', I^4, I^5, \ldots\}$ satisfy the given condition there.

Back to the question: Suppose $A$ is I-adically and J-adically complete. We will show that $A$ is $(I+J)^2$-adically complete (hence $(I+J)$-adically complete). So let $(a_1, a_2, \ldots)$ be a sequence in $A$ which satisfies $a_n - a_{n+1} \in (I+J)^2$ for every $n$.

In particular, $a_n - a_{n+1} \in (I+J)^2 \subseteq I^n + J^n$, so $a_n - a_{n+1} \in I^n + J^n$, hence $a_n$ is $I^n + J^n$-adically Cauchy.

Let $c_n = \sum_{k=1}^{\infty} c_k$ and $d_n = \sum_{k=1}^{\infty} d_k$. 


Since \( C_n - C_{n-1} = c_n \in I^n \) for every \( n \) and \( A \) is I-adically complete, \( \exists ! \) \( C \in A \) such that
\[
C - C_n \in I^n.
\]
Similarly, \( \exists ! \) \( D \in A \) such that
\[
D - D_n \in J^n.
\]
Observe that
\[
C_n + D_n = \sum_{k=1}^{\infty} (c_k + d_k)
\]
\[
l = \sum_{k=1}^{\infty} (a_k - a_{k-1})
\]
\[
= a_n - a_1.
\]
So if we let \( a = C + D + a_1 \), we have
\[
a - a_n = C + D + a_1 - (C_n + D_n + a_1)\]
\[
= (C - C_n) + (D - D_n) \in I^n + J^n \subseteq (I+J)^n.
\]

8.1 Let \( (a_n) \) be a Cauchy sequence in \( A \) w.r.t. the \( (I+J) \)-adic topology. So given \( m \in \mathbb{Z}^+ \), \( \exists N_m \) s.t.
\[
p, q > N_m \Rightarrow a_p - a_q \in (I+J)^m \subseteq I^m + J^m.
\]

In particular, \( n > N_m \Rightarrow a_{n+m} - a_n \in I^n + J^n \).

We may take \( N_1 < N_2 < N_3 < \ldots \).

So then
\[
a_{N_1} - a_{N_2} \in I + J
\]
\[
a_{N_2} - a_{N_3} \in (I+J)^2
\]
\[
a_{N_3} - a_{N_4} \in I^2 + J^2
\]
\[
\vdots
\]

That is, \( (a_n) \) is Cauchy. (Note: \((I+J)\)-adic topology is Hausdorff, hence metrizable because \( A \) is I-adically complete, \( I \subseteq \text{rad}(A) \) so \( I+J \subseteq \text{rad}(A) \) hence \( (I+J)^n = 0 \) by Krull's Intersection Theorem.)
Let's ease up the notation. What we showed above is that 
\((a_n)\) has a subsequence \((a_{n_k}) = (b_n)\) such that

\[ b_n - b_{n+1} \in I^n J^n. \quad (\star) \]

A Cauchy sequence with a convergent subsequence must be convergent itself, so it is enough to show that \((b_n)\) converges.

By \((\star)\), there exist \(u_n, v_n \in I^n\) such that

\[ b_{n+k} - b_n = u_n + v_n. \quad (\dagger) \]

Now let \(U_n = \sum_{k=1}^{n} u_k\) and \(V_n = \sum_{k=1}^{n} v_k\). Observe that \((U_n)\) is a Cauchy sequence in the \(I\)-adic topology

\((\text{because for } \rho < q, \quad U_q - U_p = \sum_{k=p}^{q} u_k \in I^\rho)\).

Since \(A\) is \(I\)-adically complete, \((U_n)\) has a limit, say \(U\).

By the same argument, \((V_n)\) has a limit in the \(J\)-adic topology, say \(V\).

Let \(B_n = U_n + V_n + 1\). Observe that

\[ \sum_{k=1}^{n} (b_{n+k} - b_k) = \sum_{k=1}^{n} (u_k + v_k) \]

\(\downarrow\) telescope

\[ b_n - b_1 = U_n + V_n. \]

So we write

\[ b_n = U_n + V_n + 1. \]

Now note that \(I\)-adic convergence is stronger than \((I+J)\)-adic convergence. So \(U_n \rightarrow U\) in the \((I+J)\)-adic topology as well. Similarly, \(V_n \rightarrow V\) in the \((I+J)\)-adic topology. Thus \(b_n \rightarrow U + V + 1\) in the \((I+J)\)-adic topology.

We found a limit for \((b_n)\).
8.2 Since $A$ is $I$-adically complete, $I \subseteq \operatorname{rad} A$. Hence $J \subseteq \operatorname{rad} A$ and therefore the $J$-adic topology on $A$ is Hausdorff by Krull's intersection theorem.

Now let $(a_n)$ be a Cauchy sequence in $A$ w.r.t $J$-adic topology. Then $(a_n)$ is also a Cauchy sequence in $A$ w.r.t $I$-adic topology (the $J$-adic topology is finer), thus converges to an element $a$ in the $I$-adic topology.

We want to show that $a_n \to a$ in the $J$-adic topology also.

By passing to a subsequence if necessary, as we did in $(8.1)$, we may assume that

$$a_{n+1} - a_n \in I^n \quad \text{for } n \geq N.$$ 

Also for sufficiently large $m, n$ we have $a - a_m \in I^m$.

Then

$$a - a_n = a - a_m + a_m - a_n \quad \in I^m + J^n.$$ 

By telescoping

$$
\sum_{n \geq N} (a_{n+1} - a_n) \in J^n,
\sum_{n \geq N} (a_{n+1} - a_n) \in J^n
$$

Then $a - a_n \in \bigcap_{m \geq N} (I^m + J^n)$. By Theorem 8.7, the finite $A$-module $A/\mathfrak{p}$ is $I$-adically complete. Thus $\bigcap_{m \geq N} I^n (A/\mathfrak{p}) = \bigcap_{m \geq N} (I^m + J^n)/\mathfrak{p} = 0$, i.e. $\bigcap_{m \geq N} (I^m + J^n) = J^n$. Thus $a - a_n \in J^n$. This yields $a_n \to a$ in $J$-adic topology, as desired.