Let $f(z) = z^2 + z$. Note that

$$f(z) = f(w) \implies z^2 + z = w^2 + w$$
$$\implies z^2 - w^2 + z - w = 0$$
$$\implies (z-w)(z+w+1) = 0$$
$$\implies z = w \text{ or } z+w = -1.$$

So in the disk $D(0; 1/2)$ $f$ is one to one because for $z, w \in D(0; 1/2)$, $|z+w| \leq |z| + |w| < 1/2 + 1/2 = 1$ so $z+w 
eq -1$.

We claim that $D(0; 1/2)$ is the largest such disk around the origin. Indeed, consider a larger disk $D(0; r)$ with $r > 1/2$.

Let $\varepsilon = r - 1/2 > 0$. New $-1 + \varepsilon/2, -1 + \varepsilon/2$ are distinct points and both lie in $D(0; r)$ because

$$\left| -1 - \frac{\varepsilon}{2} \right| = \left| 1 + \frac{\varepsilon}{2} \right| = \frac{1}{2} + \frac{\varepsilon}{2} < \frac{1}{2} + \varepsilon = r$$
$$\left| -1 + \frac{\varepsilon}{2} \right| \leq \left| -1 \right| + \left| \frac{\varepsilon}{2} \right| = \frac{1}{2} + \frac{\varepsilon}{2} < r.$$

However, $f\left(-1 - \frac{\varepsilon}{2}\right) = f\left(-1 + \frac{\varepsilon}{2}\right)$ since $-1 - \frac{\varepsilon}{2} + -1 + \frac{\varepsilon}{2} = -1$.

Let $f(z) = e^z$. We have

$$f(z) = f(w) \implies e^z = e^w \implies e^{z-w} = 1 \implies z - w = n \cdot 2\pi i \text{ for some } n \in \mathbb{Z}.$$

Therefore $f$ is one to one in the disk $D(0; 3\pi)$ because for $z, w \in D(0; 3\pi)$ $|z-w| < 2\pi$, so $f(z) = f(w) \implies 2\pi > |z-w| = |z| - |w| > 3\pi - 3\pi = 0$ for some $n \in \mathbb{Z}$ have $n = 0$.
And \( D(0, π) \) is the largest such disk because for \( r > π \), 
\( iπ, -iπ \in D(0, r) \) and \( e^{iπ} = e^{-iπ} = -1 \).

\[ f(z) = \cos z \cos 0 = 1, \] so let \( F(z) = f(z) - 1 = \cos z - 1 \)

Note that \( F \) is given by the power series

\[ F(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \]

We see that \( F \) has a zero of order 2 at 0, and

\[ F(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+2}}{(2n+2)!} = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+2)!} \]

so \( \lim_{z \to 0} \frac{F(z)}{z^2} = \frac{-1}{2} \)

Thus \( f(z) - 1 = F(z) = z^2 g(z) \) where \( g \) is the analytic function given by

\[ g(z) = \begin{cases} 
\frac{f(z) - 1}{z^2}, & z \neq 0 \\
-\frac{1}{2}, & z = 0
\end{cases} \]

Note that \( \cos z = 1 - 2 \sin^2 \left( \frac{z}{2} \right) \), so

\[ g(z) = \begin{cases} 
-2 \sin^2 \left( \frac{z}{2} \right), & z \neq 0 \\
-\frac{1}{2}, & z = 0
\end{cases} \]

So \( g(z) = \frac{\zeta(z)^2}{z^2} \) where \( \zeta(z) \) is the analytic function given by

\[ \zeta(z) = \begin{cases} 
\frac{i \sqrt{2} \sin \left( \frac{z}{2} \right)}{z}, & z \neq 0 \\
\frac{i}{\sqrt{2}}, & z = 0
\end{cases} \]

So for \( \zeta(z) = z \cdot h(z) \), we get \( \cos z = f(z) = 1 + \zeta(z)^2 \)
Let \( D \) be the open unit disk as above and let
\[
D = \{ z \in \mathbb{C} : \Re(z) > 0 \}
\]
be the upper half-plane.

Let \( a \in D \). Since \( \overline{a} \) is symmetric to \( a \) with respect to the real axis, we have
\[
z \in D \iff z \text{ is closer to } a \text{ than } \overline{a} \iff |z - a| < |z - \overline{a}|
\]
\[
\iff \left| \frac{z - a}{z - \overline{a}} \right| < 1.
\]

So \( T_a(z) = \frac{z - a}{z - \overline{a}} \) is a linear transformation such that
\[
T_a(D) = D.
\]

To prove this, we can regard \( T_a \) as an analytic function \( T_a : D \to D \).

Note that \( T_a(\overline{a}) = 0 \).

By what is given in the question, we can write that
\[
f : D \to \mathbb{C}
\]
is an analytic function with \( f(D) \subseteq D \).

If \( f \) is constant, clearly
\[
\left| \frac{f(z) - f(a)}{f(a) - f(a)} \right| = 0 \leq \left| \frac{z - a}{z - \overline{a}} \right| \quad \text{for any } z \in D.
\]

So we can assume \( f \) is nonconstant. Then by the open mapping theorem \( f(D) \) is open in \( \mathbb{C} \) so \( f(D) \subseteq \text{Int}(D) = D \). Hence we can regard \( f \) as an analytic function \( f : D \to D \).

Let \( a \in D \) and \( b = f(a) \in D \). Consider the composition
\[
g : D \xrightarrow{T_a^{-1}} D \xrightarrow{f} D \xrightarrow{T_b} D
\]
So \( g : D \to D \) is an analytic function with
\[
g(0) = (T_b \circ f \circ T_a^{-1})(0) = (T_b \circ f)(a) = T_b(b) = 0.
\]
Hence by the Schwarz lemma
\[ |g(w)| \leq |w| \text{ for all } w \in \mathbb{D}, \]

In particular
\[ |g(T_a(z))| \leq |T_a(z)| \]
\[ |T_b(f(z))| \leq |T_a(z)| \]
\[ \left| \frac{f(z) - b}{f(z) - a} \right| = \left| \frac{f(z) - b}{f(z) - a} \right| = \left| \frac{z - a}{z - a} \right|, \]

We also have
\[ 1 = |g'(0)| = \left| \left( T_a^{-1}(0) \cdot f'(a) \cdot T_b'(b) \right) \right| \]
\[ = \left| \frac{1}{T_a'(a)} \cdot f'(a) \cdot T_b'(b) \right|, \]

Since
\[ T_a'(z) = \frac{d}{dz} \left( \frac{z - a}{z - a} \right) = \frac{z - a - (z - a)}{(z - a)^2} = \frac{a - a}{(z - a)^2}, \]
\[ T_a'(a) = \frac{1}{a - a} = \frac{2}{\text{Im} a}, \]

Thus,
\[ \left| T_a'(a) \right| \geq |f'(a)| \left| T_b'(b) \right| \]
\[ \frac{2}{\text{Im} a} \geq |f'(a)| \cdot \frac{2}{\text{Im} b} \]
\[ \frac{1}{\text{Im} a} \geq \frac{|f'(a)|}{\text{Im} f(a)} \text{ for every } a \in \mathbb{D}. \]
Both in Ex. 1 and Ex. 2, we found two linear transformations $T, S$ such that $S \circ f \circ T$ maps the open unit disk $U$ to itself and fixes the origin. The obtained inequality then is just by the Schwarz lemma applied to $g = S \circ f \circ T$. Therefore, if we have equality in any of these, that implies again by the Schwarz lemma that $g$ is a linear transformation (actually a rotation). Hence

$$f = S^{-1} \circ g \circ T^{-1}$$

is also a linear transformation.

We are given that $f: U \to \mathbb{H}$ is analytic. For any $a \in \mathbb{H}$, since $f(a) = b \in \mathbb{H}$ we described in Ex 2.

Then $T_b \circ f: U \to U$ is analytic. So by Ex. 1

$$\left| \frac{(T_b \circ f)'(w)}{1 - \left| (T_b \circ f)(w) \right|^2} \right| < \frac{1}{1 - \left| w \right|^2}$$

for all $w \in \partial U$.

In particular for $w = a$, we get

$$\left| \frac{T_b'(b) \cdot f'(a)}{1 - \left| T_b(b) \right|^2} \right| < \frac{1}{1 - \left| a \right|^2}$$

$$\frac{2 |f'(a)|}{\text{Im } f(a)} = 2 \frac{|f'(a)|}{\text{Im } b} < \frac{1}{1 - \left| a \right|^2}.$$
For the other inequality (whatever that means), let $a \in \mathbb{H}$. Consider $T_a : \mathbb{H} \to \mathbb{H}$, then $f \circ T_a : \mathbb{H} \to \mathbb{H}$ is analytic.

Hence by Ex. 2

$$\left| \frac{(f \circ T_a)(w) - (f \circ T_a)(a)}{(f \circ T_a)(w) - (f \circ T_a)(a)} \right| \leq |T_a(w)|$$

for all $w \in \mathbb{H}$.

So for every $z \in \mathbb{D}$, putting $w = T_a^{-1}(z)$ we get

$$\left| \frac{f(z) - f(a)}{f(z) - f(a)} \right| \leq |z|$$

913b, 5: First we investigate $\text{Aut}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{D} \mid f \text{ is analytic and has an analytic inverse} \}$

Let $f \in \text{Aut}(\mathbb{D})$ with inverse $g : \mathbb{D} \to \mathbb{D}$. By Ex. 1 for all $z \in \mathbb{D}$ we have

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

Hence for all $w \in \mathbb{D}$, putting $z = g(w)$ we get

$$\frac{|f'(g(w))|}{1 - |g(w)|^2} \leq \frac{1}{1 - |w|^2}$$

Since $|f'(g(w))| = \frac{1}{g'(w)}$, we have

$$\frac{1}{1 - |w|^2} \leq \frac{|g'(w)|}{1 - |g(w)|^2} \quad (\ast)$$

Now Ex. 1 applied to $g$ tells us that $(\ast)$ must be an equality. Then Ex. 3 gives that $g$ is a linear transformation, thus so is $f$. 