Let $10 = \{ z \in \mathbb{C} : |z| < 1 \}$. For every $a \in 10$, by the problem on p.11, 1 of the 1st homework we have a holomorphic function

$$
\Phi_a : 10 \to 10
$$

$$
z \mapsto \frac{a-z}{1-\overline{a}z}
$$

We make the following observations about $\Phi_a$:

- $\Phi_a$ is a linear transformation with matrix form $A = \begin{bmatrix} -1 & a \\ \overline{a} & 1 \end{bmatrix}$. 

Since $A^2 = \begin{bmatrix} 1-|a|^2 & 0 \\ 0 & 1-|a|^2 \end{bmatrix}$, we set $\Phi_a^2 = \text{id}$. 

In other words, $\Phi_a$ is an involution.

- $\Phi_a(a) = 0$ and $\Phi_a(0) = a$.
- $\Phi_a'(z) = \frac{d}{dz} \left( \frac{z-a}{\overline{a}z-1} \right) = \frac{\overline{a}z-1-\overline{a}(z-a)}{(\overline{a}z-1)^2}$

$$
= \frac{|a|^2-1}{(\overline{a}z-1)^2}.
$$

So $\Phi_a'(0) = |a|^2-1$ and $\Phi_a'(a) = \frac{1}{|a|^2-1}$.

The assumptions of the problem say that $f : 10 \to \mathbb{C}$ is an analytic function with $f(10) \subseteq \overline{10}$. Therefore if $f$ is constant, $f \equiv 0$ so the desired inequality trivially holds. So we may assume $f$ is nonconstant. Then by the open mapping theorem $f(10)$ is open, therefore $f(10) \subseteq \text{Int}(\overline{10}) = 10$.

So we may regard $f$ as an analytic function

$$
f : 10 \to 10.
$$
Fix $a \in \mathbb{D}$. Let $b = f(a) \in \mathbb{D}$. Consider the composition

$$g = \phi_b \circ f \circ \phi_a : \mathbb{D} \to \mathbb{D}$$

Since $g(0) = \phi_b(f(\phi_a(0))) = \phi_b(f(a)) = \phi_b(b) = 0$, and $g : \mathbb{D} \to \mathbb{D}$ is analytic, by the Schwarz Lemma,

$$1 > |g'(0)| = |\phi_b'(b) \cdot f'(a) \cdot \phi_a'(0)|$$

$$= \left| \frac{1}{|b|^2 - 1} \cdot f'(a) \cdot (|a|^2 - 1) \right|$$

$$= \frac{1}{1 - |b|^2} \cdot |f'(a)| \cdot (1 - |a|^2)$$

$$= \frac{|f'(a)|}{1 - |f(a)|^2} \cdot (1 - |a|^2)$$

Hence

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}$$

Since $a \in \mathbb{D}$ above was arbitrary, we are done.
So elements of $\text{Aut}(\Omega)$ are linear transformations. For the general case as in the question, let $f$ be a conformal mapping of $\Omega$ onto $\Omega$, where $\Omega \in \Delta$ be generalized disks in $\mathbb{C}$.

Hence we can write $f: \Omega \to \Omega$ is analytic. Since $f$ is one-to-one and onto, it has a set-theoretic inverse $f^{-1}: \Omega \to \Omega$.

By the open mapping theorem $f^{-1}$ is continuous. Moreover as $f$ is conformal, $f'$ never vanishes. Thus, together with the continuity of $f'$ implies that $f^{-1}$ is differentiable.

Now we know that there exist linear transformations (10-Unit disk)

$$T: \Omega \to \Omega$$

$$S: \Omega \to \Omega$$

Hence $gS\circ fT \in \text{Aut}(\Omega)$. By above $g$ is a linear transformation, thus so is $f = S^{-1}g \circ T^{-1}$.

---

Problem Set:

Problem Set:

5. First we observe that the relation "being homotopic in $\Delta^n(\omega)$" is a transitive relation among closed curves in $\Delta^n$ (it is actually an equivalence relation). Suppose $\gamma \sim \delta$ and $\delta \sim \psi$.

Let $\Gamma: [0,1] \times [0,1] \to \Delta^n$ be a homotopy from $\gamma$ to $\delta$ and $\Lambda: [0,1] \times [0,1] \to \Delta^n$ be a homotopy from $\delta$ to $\psi$.

Then define

$$H: [0,1] \times [0,1] \to \Delta^n$$

$$(s, t) \mapsto \begin{cases} \Gamma(s, 2t), & 0 \leq t \leq \frac{1}{2} \\ \Lambda(s, 2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$
H is well-defined since 
\[ H(s, 2 \cdot \frac{1}{2}) = H(s, 1) = \delta(s) \]
\[ = \Delta(s, 0) = \Delta(s, 2 \cdot \frac{1}{2}) \]
Moreover \( H \) is continuous since \( [0,1] \times [0,1] = \overbrace{[0,1] \times [0, \frac{1}{2}]}^{A} \cup \underbrace{[0,1] \times [\frac{1}{2}, 1]}_{B} \)
\( A, B \) are closed subsets of \( [0,1] \times [0,1] \) and \( H|_A, H|_B \) are continuous.
Finally, we observe that (by using the fact that \( \Gamma, \Delta \) are homotopies)
\[ H(s, 0) = \Gamma(s, 0) = \gamma(s) \] for all \( s \in [0,1] \).
\[ H(s, 1) = \Delta(s, 1) = \psi(s) \]
\[ H(0, t) = \begin{cases} 
\Gamma(0, 2t), & 0 \leq t \leq \frac{1}{2} \\
\Delta(0, 2t-1), & \frac{1}{2} \leq t \leq 1
\end{cases} \]
\[ = \begin{cases} 
\Gamma(1, 2t), & 0 \leq t \leq \frac{1}{2} \\
\Delta(1, 2t-1), & \frac{1}{2} \leq t \leq 1
\end{cases} \]
Thus \( \gamma \sim \psi \).

Hence for the question, it suffices to show that for any \( a, b \in \mathcal{L} \) we have \( \gamma_a \sim \gamma_b \).
Indeed, since \( \mathcal{L} \) is open and connected, it is path-connected. So there exists continuous \( \lambda : [0,1] \to \mathcal{L} \) with \( \lambda(0) = a, \lambda(1) = b \).
Now define
\[ \Gamma : [0,1] \times [0,1] \longrightarrow \mathcal{L} 
\begin{align*}
(s, t) & \longmapsto \lambda(t) 
\end{align*} \]
\( \Gamma \) is clearly well-defined and continuous. Finally, we check
\[ \Gamma(s, 0) = \lambda(0) = \gamma_a(s) \text{ for all } s \in [0,1], \]
\[ \Gamma(s, 1) = \lambda(1) = \gamma_b(s) \text{ for all } s \in [0,1]. \]
\[ \Gamma(0, t) = \lambda(t) = \Gamma(1, t) \text{ for all } t \in [0,1], \]
\[ \Gamma \text{ is a homotopy from } \gamma_a \text{ to } \gamma_b. \]
2. Consider the continuous function
\[ \Gamma: [0,1] \times [0,1] \rightarrow \mathbb{C} - \{0\} \]
\[ (s,t) \mapsto e^{2\pi i s(1-t)} \]

We have

Then in the altered sense, the curve

Let \( \gamma: [0,1] \rightarrow \mathbb{C} - \{0\} \)
\[ s \mapsto e^{2\pi i s} \]

\( c: [0,1] \rightarrow \mathbb{C} - \{0\} \)
\[ s \mapsto 1 \]

\( \gamma \) and \( c \) are closed curves in \( \mathbb{C} - \{0\} \) and

1) \( \Gamma(s,0) = e^{2\pi i s} = \gamma(s) \)
2) \( \Gamma(s,1) = e^0 = 1 = c(s) \)

for all \( s \in [0,1] \)

Hence \( \gamma \) and \( c \) are "homotopic" in the altered sense. However,

\[ \int_{\gamma} \frac{1}{z} \, dz = 2\pi i \neq 0 \neq \int_{c} \frac{1}{z} \, dz . \]