First we verify some general statements about residues. Let \( f \) have a pole of order \( N \) at \( z_0 \).

Then \( g(z) = (z-z_0)^N f(z) \) has a removable singularity at \( z_0 \). So if \( \sum_{n=0}^\infty a_n (z-z_0)^n \) is the local power representation of \( f \) around \( z_0 \), the residue \( c_{-1} \) is the coefficient of \( (z-z_0)^{N-1} \) in the power series for \( g \) around \( z_0 \).

Thus

\[
\text{Res} \left( f, z_0 \right) = c_{-1} = \frac{g^{(N-1)}(z_0)}{(N-1)!}
\]

As a corollary, if \( g \) is analytic around \( z_0 \) and

\[
f(z) = \frac{g(z)}{(z-z_0)^N}
\]

then \( f \) has a pole of order \( N \) at \( z_0 \) and by above

\[
\text{Res} \left( f, z_0 \right) = \frac{g^{(N-1)}(z_0)}{(N-1)!}
\]

In particular, if \( N=1 \) \( \text{Res} \left( f, z_0 \right) = g(z_0) = \frac{1}{z_0+z_0} f(z_0) \).

a) Note that \( z^{8} - 1 = (z^4 - 1)(z^4 + 1) \), so the roots of \( z^4 + 1 \) are the 8th roots of unity and they are distinct from each other. Let \( \zeta = e^{2\pi i/8} \), \( \zeta \) is a primitive 8th root of unity, hence \( \zeta^4 \neq 1 \). Thus \( \zeta \) is a root of \( z^4 + 1 \). Since all the roots are powers of \( \zeta \), it is easy to see that the roots of \( z^4 + 1 \) are \( \zeta, \zeta^3, \zeta^5, \zeta^7 \).

Thus these are the singularities of \( f(z) = \frac{z}{z^4 + 1} \) and they are all simple poles (since they are simple non-repeated roots of \( z^4 + 1 \)).
\[
\text{Thus } \text{Res}(f, \pi) = \lim_{x \to \pi} (x - \pi) f(x) = \frac{\pi}{(2 - \pi)(3 - \pi)(4 - \pi)}
\]
\[
= \lim_{x \to \pi} (x - \pi) \cdot \frac{\pi}{(2 - \pi)(3 - \pi)(4 - \pi)}
\]
\[
= \frac{\pi}{(2 - \pi)(3 - \pi)(4 - \pi)}
\]

Similarly \[
\text{Res}(f, \pi^3) = \frac{\pi^3}{(\pi^3 - \pi)(\pi^3 - \pi^2)(\pi^3 - \pi^3)}
\]
\[
\text{Res}(f, \pi^5) = \frac{\pi^5}{(\pi^5 - \pi)(\pi^5 - \pi^4)(\pi^5 - \pi^5)}
\]
\[
\text{Res}(f, \pi^7) = \frac{\pi^7}{(\pi^7 - \pi)(\pi^7 - \pi^6)(\pi^7 - \pi^7)}
\]

\[b) \quad f(X) = \frac{\sin X}{X^2 (\pi - X)} \quad \text{let } \text{g}(X) = \frac{\sin X}{X^2}
\]

\[\text{When } \pi \text{ is a simple root of the denominator polynomial, } \sin X \text{ is analytic everywhere. So } \pi \text{ is a simple pole of } f.
\]

Thus \[
\text{Res}(f, \pi) = \lim_{x \to \pi} (x - \pi) f(x) = \lim_{x \to \pi} (x - \pi) \cdot \frac{\sin X}{X^2 (\pi - X)}
\]
\[
= -\frac{\sin \pi}{\pi^2} = 0
\]

Let \[
\text{g}(X) = \frac{\sin X}{X^2}
\]

Then \(g\) is analytic around 0 and \[
\text{g}(X) = \frac{\sin X}{X^2} \quad \text{hence} \quad \text{Res}(f, 0) = \frac{\text{g}'(0)}{1!} = \text{g}'(0)
\]

Since \[
\text{g}'(X) = \frac{\cos X (\pi - X) - \sin X (-1)}{(\pi - X)^2} \quad \text{g}'(0) = \frac{1 \cdot \pi - 0}{\pi^2} = \frac{\pi}{\pi^2} = \frac{1}{\pi}
\]
0, \pi are the only roots of the denominator, so if has no other singularities.

c) \( f(z) = \frac{z e^{iz}}{(z - \pi)^2} \). Let \( g(z) = z e^{iz} \). \( g \) is analytic around \( \Pi \)

and \( f(z) = \frac{g(z)}{(z - \pi)^2} \). Hence \( \text{Res}(f, \pi) = g'(\pi) \).

Since \( g'(z) = e^{iz} + z e^{iz} \), \( g'(\pi) = e^{i\pi} + \pi e^{i\pi} = 1 - i\pi \).

No other singularities. The denominator only has \( \pi \) as a root.

d) \( f(z) = \frac{z^3 + 5}{(z^4 - 1)(z + 1)} \). Let \( \tilde{f}(z) = \frac{z^3 + 5}{z^4 - 1} \).

Then clearly \( f \) and \( \tilde{f} \) have the same singularities. Moreover if \( z_0 \) is a singularity of \( f \),

\( \text{Res}(f, z_0) = \text{Res}(\tilde{f}, z_0) \)

because the \(-1^{st}\) term of the power series representations of \( f \) and \( \tilde{f} \) will be the same.

Now, the singularities of \( f \) are the roots of \( z^4 - 1 \), which are 6th roots of unity: \( \pm 1, \pm i \). These are simple roots of \( z^4 - 1 \),

hence are simple poles of \( f \) and \( \tilde{f} \). \( (z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i)) \)

Thus \( \text{Res}(f, 1) = \text{Res}(\tilde{f}, 1) = \lim_{z \to 1} (z - 1) \cdot \frac{z^3 + 5}{z^4 - 1} = \frac{3}{2} \).

Similary

\( \text{Res}(f, -1) = \frac{(-1)^3 + 5}{(-1 - 1)(-1 - i)(-1 + i)} \cdot (-1) = \frac{4}{(-2)(-1)i} = \frac{4}{2} = 2 \).

Therefore

\( \text{Res}(f, 1) = \frac{3}{2} \) and \( \text{Res}(f, -1) = 2 \).
\[
\text{Res}(f, i) = \frac{(i^3 + 5)}{(i-1)(i+1)(i+1)} \cdot i = \frac{-i + 5}{(i^2 - 1) \cdot 2i} \cdot i = \frac{5 - i}{(-2) \cdot 2}
\]

\[
\therefore \text{Res} = \frac{-5 + i}{4}
\]

\[
\text{Res}(f, -i) = \frac{(-i)^3 + 5}{(-i-1)(-i+1)(-i+1)} \cdot (-i) = \frac{i + 5}{(-4)(i+i)(i-i) \cdot 2}
\]

\[
= \frac{i + 5}{(-4) \cdot 2 \cdot 2} = \frac{-5 - i}{4}
\]

2. a) Let \( f(z) = \frac{e^{\pi z}}{4z^2 + 1} \). The numerator is analytic everywhere.

The denominator \( 4z^2 + 1 \) has two simple roots \( \frac{i}{2}, -\frac{i}{2} \). So these are the only singularities of \( f \) and they are simple poles.

We have, then (by (1))

\[
\text{Res}(f, i/2) = \lim_{z \to i/2} (z - i/2) \cdot f(z)
\]

\[
= \lim_{z \to i/2} \frac{e^{\pi z}}{(2z - i)(2z + i)}
\]

\[
= \lim_{z \to i/2} \frac{e^{\pi z}}{2 \cdot (2z + i)}
\]

\[
= \frac{e^{\pi/4}}{2 \cdot (i + i)} = \frac{1}{2 \cdot 2i} = \frac{i}{8i} = \frac{1}{4}
\]

Similarly,

\[
\text{Res}(f, -i/2) = \lim_{z \to -i/2} (z + i/2) \cdot f(z)
\]

\[
= \lim_{z \to -i/2} \frac{e^{\pi z}}{(2z - i) \cdot 2} = \frac{e^{-\pi/4}}{(-2i) \cdot 2} = \frac{-i}{-4i} = \frac{1}{4}
\]

By the residue theorem with \( \Gamma = C \),

\[
\frac{1}{2\pi i} \int_{\Gamma} f(z)\,dz = n(C, i/2) \cdot \text{Res}(f, i/2) + n(C, -i/2) \cdot \text{Res}(f, -i/2) = \frac{1}{4} + 1 \cdot \frac{1}{4} = \frac{1}{2}
\]

so \( \int_{\Gamma} f(z)\,dz = 4\pi i \).
b) Let \( f(t) = \frac{e^t}{(t^2 + t - 3/4)^2} = \frac{e^t}{(t - 1/2)^2 (t + 3/2)^2} \)

So the only singularities of \( f \) are \( 1/2 \) and \(-3/2\). They are both \( 2 \)nd order poles.

Applying the residue theorem for \( \mathbb{R} = \mathbb{C} \),

\[
\int_C f(t) \, dt = 2\pi i \left( \operatorname{Res}(f, 1/2) + \operatorname{Res}(f, -3/2) \right)
\]

\[
= 2\pi i \, \operatorname{Res}(f, 1/2).
\]

Let \( g(t) = \frac{e^t}{(t + 3/2)^2} \). Then \( g \) is analytic around \( 1/2 \) and

\[
\frac{g(t)}{(t - 1/2)^2}. \quad \text{Hence} \quad \operatorname{Res}(f, 1/2) = g'(1/2).
\]

Since \( g'(t) = \frac{e^t (t + 3/2)^2 - e^t \cdot 2(t + 3/2)}{(t + 3/2)^4} \),

\[
g'(1/2) = \frac{e^{1/2} \cdot 2^2 - e^{1/2} \cdot 2 \cdot 2}{2^4} = 0
\]

So \( \operatorname{Res}(f, 1/2) = 0 \) and hence \( \int_C f(t) \, dt = 0 \).
3. With the given substitution(s),
\[
\frac{1}{2 + \sin \theta} = \frac{1}{2 + \frac{e^{i \theta} - 1}{2i}e^{-i \theta}} = \frac{2i \frac{e^{i \theta} + 1}{2}}{e^{i \theta} - 1}
\]

And
\[
\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \int_C \frac{2i \frac{e^{i \theta} + 1}{2}}{e^{i \theta} - 1} d\theta = \int_C \frac{dz}{z^2 + 4iz - 1}
\]

Let \( f(z) = \frac{1}{z^2 + 4iz - 1} \)

Finding the roots of the denominator:
\[
\Delta = (4i)^2 - 4(-1) - 1 = -16 + 4 = -12
\]
\[
z_{1,2} = \frac{-4i \pm \sqrt{-12}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2} = \frac{-2i \pm \sqrt{3}i}{2}
\]

So the singularities of \( f \) are \( z_1 = -2 + \sqrt{3}i \) and \( z_2 = -2 - \sqrt{3}i \), both are simple poles.

Hence
\[
\text{Res} \left( f, \frac{1}{z_1} \right) = \lim_{z \to z_1} (z - z_1) \cdot f(z) = \lim_{z \to z_1} \frac{1}{z - z_2}
\]
\[
= \frac{1}{z_1 - z_2} = \frac{1}{(-2 + \sqrt{3}i, -2 - \sqrt{3}i) i} = \frac{1}{2\sqrt{3}i} = \frac{-i}{2\sqrt{3}}
\]

Similarly, \( \text{Res} \left( f, \frac{1}{z_2} \right) = \frac{1}{z_2 - z_1} = \frac{i}{2\sqrt{3}} \)

Therefore
\[
\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \int_C f(z) \, dz = 2\pi i \left( \text{Res} \left( f, \frac{1}{z_1} \right) + \text{Res} \left( f, \frac{1}{z_2} \right) \right)
\]

Residue Theorem
where \( D \) is an open ball containing \( C \).

\[
|z_1| = 2 + \sqrt{3} < 1
\]
\[
|z_2| = 2 + \sqrt{3} > 1
\]

\[
= 2\pi i \left( \frac{i}{2\sqrt{3}} + 0 \cdot \frac{i}{2\sqrt{3}} \right) = -\frac{\pi i^2}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}
\]
4. p. 154, 1. Let \( g(z) = z^7 - 2z^5 + 6z^3 - z + 1 \) and \( f(z) = 6z^3 + 1 \). \( f, g \) are entire functions.

Let \( Y \) be the unit circle. For all \( z \in \mathbb{C} - Y \), \( n(Y, z) = 0 \) or 1.

Also
\[ \{ z \in \mathbb{C} - Y : n(Y, z) = 1 \} = \{ z \in \mathbb{C} : |z| < 1 \} \]

Now, for \( z \in \mathbb{C}, \) i.e. when \( |z| = 1 \),
\[ |f(z) - g(z)| = |z^7 - 2z^5 - z| = |z| |z^6 - 2z^4 - 1| \]
\[ = |z^6 - 2z^4 - 1| \]
\[ \leq |z|^6 + 2|z|^4 + 1 = 4. \]

And
\[ |f(z)| = |6z^3 + 1| \geq |6z^3| - 1 - 1| = |6z^3| - 1| = 5 \]

Thus, for \( z \in \mathbb{C}, \) \( |f(z) - g(z)| \leq |f(z)| \).

Hence by Rouche's Theorem (where \( D = \mathbb{C} \)), \( f \) and \( g \) have the same number of roots in \( \{ z \in \mathbb{C} : |z| < 1 \} \).

Since \( f(z) = 6z^3 + 1 \)

Since \( f(z) = 0 \) \( \Rightarrow \) \( |6z^3| = 1 \)
\[ \Rightarrow |z|^3 = 1/6 \]
\[ \Rightarrow |z| < 1 , \]
every root of \( f \) is inside the unit disk. Thus \( g \) has
deg \( f = 3 \) roots inside the unit disk.
Let \( g(z) = z^4 - 6z + 3 \), \( f(z) = z^4 \), \( h(z) = -6z + 3 \).

**When \( |z| = 2 \),**

- \( |f(z) - g(z)| = |6z - 3| \leq 6|z| + 3 = 15 \)
- \( |f(z)| = |z|^4 = 16 \)

So \( |f(z) - g(z)| < |f(z)| \). Therefore \( f \) and \( g \) have the same number of roots inside \( |z| < 2 \) by Rouché's Theorem.

Obviously, \( f \) and \( g \) have 4 roots, hence so does \( g \).

**When \( |z| = 1 \),**

- \( |h(z) - g(z)| = |-z^4| = |z|^4 = 1 \)
- \( |h(z)| = |6z - 3| \geq |16z - 3| = 3 \)

So \( |h(z) - g(z)| < |h(z)| \). Therefore \( h \) and \( g \) have the same number of roots inside \( |z| < 1 \). \( h \) has a single root \( \frac{1}{2} \), which is inside \( |z| < 1 \). So \( g \) also has a single root in \( |z| < 1 \).

Thus in the region \( 1 \leq |z| < 2 \), \( g \) has 3 roots.

Since for \( |z| = 1 \) we have

\[ |g(z)| = |6z - 3 - z^4| \geq |16z - 3| - |z|^4| = |16z - 3| - 1 \]

and \( 16z - 3 \geq |16z - 3| = 3 \),

so \( |g(z)| = |16z - 3| - 1 \geq \frac{3}{2} \).

Therefore \( g \) has no root at \( |z| = 1 \).

In conclusion, \( g \) has 3 roots in the region \( 1 \leq |z| < 2 \).
b) \[ f(t) = \frac{1}{(t^2-1)^2} = \frac{1}{(t-1)^2(t+1)^2} \]

Clearly \( t = \pm 1 \) are the only poles of \( f \). They are both of order 2.

Let \( g(t) = \frac{1}{(t+1)^2} \) and \( h(t) = \frac{1}{(t-1)^2} \).

\( g \) is analytic around 1, so

\[ \text{Res} (f, 1) = \frac{g^{(2-1)}(1)}{(2-1)!} = g'(1) = (-2) (t+1)^{-3} \bigg|_{t=1} \]

\[ = \frac{-2}{2^3} = -\frac{1}{4} \]

\( h \) is analytic around -1 and \( f(t) = \frac{h(t)}{(t+1)^2} \), so

\[ \text{Res} (f, -1) = h'(1) = \frac{-2}{(t-1)^3} \bigg|_{t=-1} = \frac{-2}{(-2)^3} = \frac{1}{4} \]

e) \[ f(t) = \frac{1}{\sin^2 t} \]

Let \( g(t) = \sin t \)

For every \( k \in \mathbb{Z} \), \( g(k\pi) = 0 \) and \( g'(k\pi) = \cos (k\pi) = (-1)^k \).

Therefore each \( k\pi \) is a simple zero of \( g \). Therefore each \( k\pi \) is a pole of \( f \) of order 2.

For each \( k \in \mathbb{Z} \), let \( h_k(t) = (t-k\pi)^2 f(t) = \frac{(t-k\pi)^2}{\sin^2 t} \).

Then \( h_k \) has a removable singularity at \( k\pi \), so

\[ \text{Res} (f, k\pi) = \left( \frac{t-k\pi}{\sin t} \right)^2 \]
Then it has a removable singularity at \( k\pi \).

Indeed, since

\[
\lim_{x \to k\pi} h_k(x) = \left( \frac{2 - k\pi}{\sin x} \right)^2
\]

we have

\[
\lim_{x \to k\pi} h_k(x) = \lim_{x \to k\pi} \left( \frac{x - k\pi}{\sin x} \right)^2 = (1 - 1)^2 = 0.
\]

Moreover

\[
\cos(f, k\pi) = h_k^{-1}(k\pi) \quad \text{by the discussion in the 1st page}
\]

We know this limit exists, so we can approach \( k\pi \) on the real line

\[
\lim_{x \to k\pi} \frac{h_k(x) - h_k(k\pi)}{x - k\pi} = \lim_{x \to k\pi} \left( \frac{x - k\pi}{\sin x} \right)^2 - 1
\]

L'Hôpital's rule

\[
\lim_{x \to k\pi} 2 \left( \frac{x - k\pi}{\sin x} \right) \left( \frac{\sin x - (x - k\pi) \cos x}{\sin^2 x} \right)
\]

\[
= 2 \left( x - k\pi \right) \sin x
\]

\[
= 2 \lim_{x \to k\pi} \left( x - k\pi \right) \left( \sin x - (x - k\pi) \cos x \right)
\]

\[
= 2 \lim_{x \to k\pi} \left( \sin x - (x - k\pi) \cos x \right)
\]

\[
= 2 \sin^2 x
\]
\[
L' = 2 \lim_{x \to \pi} \frac{\sin x - (x-\pi \cos x + (\cos x - (\cos x + (x-\pi \sin x))) (x-\pi)}{3 \sin^2 x \cos x}
\]

\[
= 2 \lim_{x \to \pi} \frac{\sin x - (x-\pi) \cos x + (x-\pi)^2 \sin x}{3 \sin^2 x \cos x}
\]

\[
L' = \lim_{x \to \pi} \frac{\cos x - (\cos x + (x-\pi \sin x)) + 2(x-\pi \sin x + (x-\pi)^2 \cos x)}{3 (2 \sin x \cos^2 x + \sin^3 x)}
\]

\[
= \lim_{x \to \pi} \frac{3(x-\pi) \sin x + (x-\pi)^2 \cos x}{3 \sin x \left(2 \cos^2 x - \sin^2 x\right)}
\]

\[
= \lim_{x \to \pi} \frac{(x-\pi) \left(3 \sin x + (x-\pi) \cos x\right)}{3 \sin (x-\pi) \left(-1\right)^k \left(2 \cos^2 x - \sin^2 x\right)}
\]

Since \( \lim_{x \to \pi} \frac{x-\pi}{\sin (x-\pi)} = (-1)^k \) and \( \lim_{x \to \pi} \frac{3 \sin x + (x-\pi) \cos x}{3 \left(-1\right)^k \left(2 \cos^2 x - \sin^2 x\right)} = \frac{0}{3(-1)^k} = 0 \),
we get
\[
\text{Res} \left( f, \pi \right) = 0.
\]

**f)** \( f(z) = \frac{1}{z^m (1-z)^n} \), \( m, n \in \mathbb{Z}^+ \).

The denominator is a polynomial with a zero of order \( m \) at 0 and a zero of order \( n \) at 1. (No other zeros.)

So the poles of \( f \) are 0 and 1, and they are of order \( m \) and \( n \), respectively.

Let \( g(z) = \frac{1}{(1-z)^n} \). \( g \) is analytic around 0, and \( f(z) = \frac{g(z)}{z^m} \),

so \( \text{Res} \left( f, 0 \right) = \frac{g^{(m-1)}(0)}{(m-1)!} \).
Claim: \( \forall k \in \mathbb{N}, \quad g^{(k)}(z) = \frac{(n+k-1)!}{(n-1)!} (1-z)^{-n-k} \)

\[ g^{(k+1)}(z) = \frac{d}{dz} g^{(k)}(z) = \frac{(n+k-1)!}{(n-1)!} (-n-k)(1-z)^{-n-k-1} \cdot (-1) \]

\[ = \frac{(n+k-1)!}{(n-1)!} (n+k) \cdot (1-z)^{-n-(k+1)} \]

\[ = \frac{(n+k)!}{(n-1)!} (1-z)^{-n-(k+1)} \]

Corollary: \( g^{(k)}(0) = \frac{(n+k-1)!}{(n-1)!} \cdot \frac{1}{k!} \)

Thus \[ \text{Res}\left(f, 0\right) = \frac{(n+m-2)!}{(m-1)!} \]

\[ \text{Res} \left(f, 0\right) = \frac{1}{(m-1)!} \cdot \frac{(n+m-2)!}{(n-1)!} = \frac{(n+m-2)}{(m-1)} \]

Let \( h(z) = \frac{(-1)^n}{z^m} \). \( h \) is analytic around \( 1 \) and \( f(z) = \frac{h(z)}{(z-1)^n} \)

so \( \text{Res} \left(f, 1\right) = \frac{h^{(n-1)}(1)}{(n-1)!} \)

Claim: \( \forall k \in \mathbb{N}, \quad h^{(k)}(z) = (-1)^{n+k} \cdot \frac{(m+k-1)!}{(m-1)!} \cdot z^{-m-k} \)

Proof: Base case is again trivial.

Assuming the claim for \( k \),

\[ h^{(k+1)}(z) = \frac{d}{dz} h^{(k)}(z) = (-1)^{n+k} \cdot \frac{(m+k-1)!}{(m-1)!} (-m-k) \cdot z^{-m-k-1} \]

Corollary: \( h^{(k)}(1) = (-1)^{n+k} \cdot \frac{(m+k-1)!}{(m-1)!} \cdot \frac{1}{(m-k)!} \cdot (-1)^{m-k-1} \cdot \frac{1}{(m-k)!} \cdot z^{-m-(k+1)} \)

\[ = (-1)^{n+k} \cdot \frac{(m+k)!}{(m-1)!} \cdot \frac{1}{(m-k)!} \cdot z^{-m-(k+1)} \]
Thus, \( \text{Res}(f, 1) = \frac{1}{(n-1)!} \frac{(m-n-2)!}{(m-1)!} \).

\[ = -\binom{n+m-2}{m-1} \]

\( p.161, 3.51 \)

\( \text{c) } f(z) = \frac{z^2 - 2 + \frac{1}{2}}{z^4 + 10z^2 + 9} = \frac{\frac{2}{z^2 - 2 + \frac{1}{2}}}{(z^2 + 9)(z^2 + 1)} \]

So the poles of \( f \) are \( \pm 3i, \pm i \). They are all simple poles.

Let \( R > 3 \) and \( \gamma_R \) be the following closed curve: semicircle.

Applying the residue theorem for \( \gamma_R \), we get

\[ \frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = 2\pi i \left( \text{Res}(f, 3i) + \text{Res}(f, i) \right) \quad (\star) \]

Since simple pole

Now, \( \text{Res}(f, 3i) = \lim_{z \to 3i} (z - 3i) f(z) = \frac{z^2 - 2 + \frac{1}{2}}{(z + 3i)(z^2 + 1)} \bigg|_{z = 3i} = -\frac{9 - 3i + 2}{6i(-9 + 4)} \)

\[ = -\frac{39 - 6i}{60} = \frac{-3i + 7}{48i} = \frac{7 + 3i}{48i} \]

\( \text{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \frac{z^2 - 2 + \frac{1}{2}}{(z + i)(z^2 + 1)} \bigg|_{z = i} = -\frac{1 - i + 2}{8 \cdot 2i} = \frac{1 - i}{16i} \)

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So we can rewrite (X) as
\[ \int_{c_1} f(z) \, dz + \int_{c_2} f(z) \, dz = 2\pi i \left( \frac{7+3i}{48i} + \frac{3-3i}{48i} \right) = 2\pi i \cdot \frac{10 - 2\pi}{48i} = \frac{20\pi - 5\pi}{12} = \frac{15\pi}{12} \]

0 For \( z \in C_R, \ |z| = R \) so
\[ |z^2 - z + 2| \leq |z|^2 + |z| + 2 = R^2 + R + 2 \]
\[ |z^4 + 10z^2 + 9| = |z^4 - (-10z^2 - 9)| \geq |z^4| - |10z^2 + 9| = |R^4 - 10R^2 - 9| \text{ as } R \to \infty \]

So \( |z^4 + 10z^2 + 9| > R^4 - 10R^2 - 9 \text{ for sufficient } R \)

Thus, for \( z \in C_R, \ |f(z)| \leq \frac{R^2 + R + 2}{R^4 - 10R^2 - 9} \)

Hence
\[ \left| \int_{c_1} f(z) \, dz \right| \leq \int_{c_1} |f(z)| \, |dz| \leq \frac{R^2 + R + 2}{R^4 - 10R^2 - 9} \cdot \pi R \to 0 \text{ as } R \to \infty. \]

Therefore
\[ \frac{5\pi}{12} = \lim_{R \to \infty} \int_{c_2} f(z) \, dz = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx \]

Since we know a priori that \( \int f(x) \, dx \) converges because it grows like \( \frac{1}{x^2} \).

\( f(z) = \frac{z^2}{(z^2 + a^2)^3} \). The singular points of \( f \) are \( \mp ai, \) they are both poles of order 3. Set up the same \( y_R, \ c_R, \ iR \) as above with \( R > a \) (so \( a \) lies in \( y_R \) whereas \( ai \) does not) would be nonzero otherwise the integral diverges, WLOG we can assume \( a > 0 \).
So, similar to (c) we get
\[
\int_{C_\alpha} f(t) \, dt + \int_{I_R} f(t) \, dt = 2\pi i \left( \text{Res} \left( \frac{f}{z-a}, \alpha \right) \right)
\]

Hence, let \( g(t) = \frac{z^2}{(z+a)^3} \) be analytic around \( \alpha \).

Then \( f(t) = \frac{g(t)}{(z-a)^3} \) and hence
\[
\text{Res} \left( f, \alpha \right) = \frac{g''(\alpha)}{2!}
\]

\[
g''(t) = \frac{2z(z+a)^3 - z^2 g'(z+a)}{(z+a)^6} = \frac{2z(z+a)^3 - 3z^2}{(z+a)^4} = \frac{-z^2 + 2a z}{(z+a)^4}
\]

\[
g''(\alpha) = \frac{-((2a^2 + 2a \alpha) - 4 \cdot (2a \alpha)^3)}{(2a \alpha)^8} = \frac{-4(a^2)}{(2a \alpha)^8} = \frac{4}{32a^3} \cdot \frac{1}{i}
\]

\[
= \frac{1}{8a^3} \cdot i
\]

Thus
\[
\int_{C_\alpha} f(t) \, dt + \int_{I_R} f(t) \, dt = 2\pi i \cdot \frac{1}{2} \cdot \frac{1}{8a^3} \cdot \frac{1}{8a^3} = \frac{\pi}{8a^3}
\]

For \( \tau \in C_R \), \( |f(\tau)| = \frac{\ell^2}{(\ell^2 + \alpha^2)^3} \) and since \( (\ell^2 + \alpha^2) > \ell^2 - \ell^2 |1\alpha^2| = \ell \)

\[
|f(\tau)| \leq \frac{\ell^2}{(\ell^2 - \ell^2 |1\alpha^2|)^3}
\]

Hence
\[
\left| \int_{C_\alpha} f(t) \, dt \right| \leq \int_{C_\alpha} |f(t)| \, dt \leq \frac{\ell^2}{(\ell^2 - \ell^2 |1\alpha^2|)^3} \cdot \pi \ell \to 0 \quad \text{as} \quad \ell \to \infty
\]
Therefore, 

\[
\frac{\pi}{8a^3} = \lim_{R \to \infty} \int_{I_R} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = 2 \lim_{R \to \infty} \int_{0}^{R} f(x) \, dx \\
= 2 \int_{0}^{\infty} f(x) \, dx
\]

Hence 

\[
\int_{0}^{\infty} f(x) \, dx = \frac{\pi}{16a^3}
\]

For \( a > 0 \), WLOG we can assume \( a > 0 \), because for \( a = 0 \) the integral diverges.

Set the curves \( \gamma_R \), \( C_R \), \( I_R \) as in (c), with \( R > a \).

\( f \) has a unique singularity inside \( \gamma_R \) : \( ai \).

\( ai \) is a simple pole of \( f \), so

\[
\text{Res} (f, ai) = \lim_{t \to ai} \frac{(t - ai) \, f(t)}{t - ai} = \frac{e^{i \cdot ai}}{ai + ai} = \frac{e^{-a}}{2ai}
\]

Hence 

\[
\int_{C_R} f(t) \, dt + \int_{I_R} f(t) \, dt = 2\pi i \cdot \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}
\]

For \( z \in C_R \), \( |e^{iz}| = |e^{-z+ix}| = |e^{-y}| \cdot |e^{ix}| = e^{-y} \leq 1 \) since \( z \) is in the upper half plane.

\( |z^2 + a^2| > |z^2| - |a^2| = |z^2 - a^2| = \Re^2 - a^2 \)

So 

\[
\int_{C_R} |f(t)| \, dt = \int_{C_R} |f(t)| \, dt \leq \frac{1}{\Re^2 - a^2} \cdot \pi R \to 0 \quad \text{as} \quad R \to \infty
\]
Thus,
\[ \frac{\pi e^{-a}}{a} = \lim_{R \to \infty} \frac{1}{2} \int_{-R}^{R} f(x) \, dx \]

where
\[ f(x) = \frac{\cos x}{x^2 + 1} \]

is real. Hence,
\[ = \lim_{R \to \infty} 2 \int_{0}^{R} f(x) \, dx \]
\[ = 2 \lim_{R \to \infty} \left( \int_{0}^{R} \text{Re}(f(x)) \, dx + i \int_{0}^{R} \text{Im}(f(x)) \, dx \right) \]

By comparison with \( \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx \)
\[ = \int_{0}^{\infty} \text{Re}(f(x)) \, dx \]

so we can write
\[ \frac{\pi e^{-a}}{a} = 2 \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx + 2i \int_{0}^{\infty} \frac{\sin x}{x^2 + a^2} \, dx \]

LHS is real, so
\[ \frac{\pi e^{-a}}{a} = 2 \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx \]

\[ \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = \frac{\pi e^{-a}}{2a} \]

If \( a = 0 \), we can integrate by parts:

\[ \int_{0}^{\infty} \frac{x^{1/3}}{1 + x^2} \, dx \]

So the integral becomes
\[ \int_{0}^{\infty} 3u^2 \, du \]

9) Evaluate \( \int_{1}^{\infty} \frac{u^{1/3}}{1 + u^2} \, du \) using the change of variable \( u = x^{1/3} \), \( x = u^3, \) \( dx = 3u^2 \, du \).
9) Consider \( \text{Arg} : \mathbb{C} - \{0, \infty\} \to (0, 2\pi) \)

\[
\text{Arg} \to 0 \quad \text{such that} \quad e^{i0} = z
\]

Then \( \text{L} : \mathbb{C} - \{0, \infty\} \to \mathbb{C} \)

\[
\begin{align*}
L(z) & = \ln|z| + i \cdot \text{Arg}(z) \\
\end{align*}
\]

is a continuous function with \( e^{L(z)} = z \) for all \( z \in \mathbb{C} - \{0, \infty\} \).

Since \( L \) is cont. and \( e^{e^t} \) never vanishes, \( L \) is holomorphic.

Let \( \phi(t) = e^{\frac{t}{3}L(t)} \). Then \( \phi(t)^3 = t \).

Let \( f(z) = \frac{\phi(t)}{1 + z^2} \). Consider the contour (radius \( R, r, \) by \( \frac{1}{2n} \) width)
$\gamma_{r,\pi,\mu}$ has two simple poles $i$, $-i$ inside $\gamma_{L,R,M}$ (Take $r < 1 < R$).

$$\text{Res} (f, i) = \lim_{z \to i} (z - i) f(z) = \frac{\phi(i)}{2i} = \frac{e^{\frac{\pi}{2} i}}{2i} = \frac{e^{\frac{\pi}{2} i}}{2i} \quad (\text{Take } e^{\frac{\pi}{2} i})$$

$$\text{Res} (f, -i) = \lim_{z \to -i} (z + i) f(z) = \frac{\phi(-i)}{-2i} = \frac{e^{-\frac{\pi}{2} i}}{4i} \quad (\text{Take } e^{-\frac{\pi}{2} i})$$