The questions are from Stein and Shakarchi’s text, Chapter 2.

1. Given a collection of sets $E_1, E_2, \ldots, E_n$, construct another collection $E_1^*, E_2^*, \ldots, E_N^*$, with $N = 2^n - 1$, so that $\bigcup_{k=1}^n E_k = \bigcup_{j=1}^N E_j^*$; the collection $\{E_j^*\}$ is disjoint; also $E_k = \bigcup_{E_j^* \subseteq E_k} E_j^*$, for every $k$.

Observe that it suffices to construct such a collection with $N \leq 2^n - 1$ because in case $N < 2^n - 1$ we can pad empty sets to get to exactly $2^n - 1$ sets and the conclusions will still hold.

Let $\mathcal{C} = \{E_1, \ldots, E_n\}$. Let $\mathcal{D}$ be the collection of all subsets of $X = \bigcup_{k=1}^n E_k$ which have expressions of the type $E'_1 \cap \cdots \cap E'_n$ where $E'_i$ is either $E_i$ or $E_i^c$ for each $i = 1, \ldots, n$.

The function $f : 2^\mathcal{C} \to \mathcal{D}$

$$f(\mathcal{E}) = \bigcap_{E \in \mathcal{E}} E \cap \bigcap_{E^c \notin \mathcal{E}} E^c$$

is surjective by the definition of $\mathcal{D}$, hence $|\mathcal{D}| \leq |2^\mathcal{C}| = 2^n$. Note that if $\mathcal{E} = \emptyset$, then $f(\mathcal{E}) = E_1^c \cap \cdots \cap E_n^c = X^c = \emptyset$. So we can restrict $f$ as $f : 2^\mathcal{C} \setminus \{\emptyset\} \to \mathcal{D} \setminus \{\emptyset\}$.

First, we show that

$$X = \bigcup_{\mathcal{E} \in 2^\mathcal{C} \setminus \{\emptyset\}} f(\mathcal{E}) = \bigcup_{\mathcal{D} \in \mathcal{D} \setminus \{\emptyset\}} D. \quad (\ast)$$

The second equality directly follows from $f$ being surjective. For the first equality, let $x \in X$. If we let $\mathcal{E}_x = \{E \in \mathcal{C} : x \in E\}$, we have $x \in f(\mathcal{E}_x)$ by the definition of $f$.

Second, we show disjointness. Say $x \in f(\mathcal{E})$. Then for every $E \in \mathcal{E}$, we have $x \in E$. Hence $\mathcal{E} \subseteq \mathcal{E}_x$. Conversely whenever $E \notin \mathcal{E}$, we have $x \notin E$. Thus $\mathcal{E} = \mathcal{E}_x$. From here we deduce that

$$f(\mathcal{E}) \cap f(\mathcal{E}') \neq \emptyset \Rightarrow \mathcal{E} = \mathcal{E}'.$$

Third, for $E \in \mathcal{C}$ we claim that

$$E = \bigcup_{\mathcal{E} \in 2^\mathcal{C} \setminus \{\emptyset\}} f(\mathcal{E}) = \bigcup_{\mathcal{D} \in \mathcal{D} \setminus \{\emptyset\}} D.$$

Again the second equality is immediate. For the first equality the $\supseteq$ part is trivial. Let $x \in E$, so $E \in \mathcal{E}_x$, equivalently $E \in \mathcal{E}_x$, and $x \in f(\mathcal{E}_x)$ so we are done.

Thus writing $\mathcal{D} \setminus \{\emptyset\} = \{E_1^*, \ldots, E_N^*\}$, we have $N \leq 2^n - 1$ with the desired properties.

2. In analogy to Proposition 2.5, prove that if $f$ is integrable on $\mathbb{R}^d$ and $\delta > 0$, then $f(\delta x)$ converges to $f(x)$ in the $L^1$-norm as $\delta \to 1$. 
Since the set of continuous functions with compact support is dense in \( L^1(\mathbb{R}^d) \), given \( \varepsilon > 0 \) there exists such a function \( g \) such that \( \|f - g\| \leq \varepsilon \). Writing \( f_\delta \) for the function \( x \mapsto f(\delta x) \), we have

\[
\|f_\delta - g_\delta\| = \|(f - g)_\delta\| = \delta^{-d}\|f - g\|.
\]

Since \( g \) is uniformly continuous on \( K := \text{supp} \ g \), there exists \( \lambda > 0 \) such that \( |g(x) - g(y)| \leq \varepsilon/m(K) \) whenever \( |x - y| \leq \lambda \) and \( x, y \in K \). Since \( K \) is compact, there exists \( M > 0 \) such that \( |x| \leq M \) for every \( x \in K \). Choosing \( \delta \) close enough to 1, we can guarantee that \( |1 - \delta| \leq \lambda/M \). Therefore for every \( x \in K \) we get

\[
|x - \delta x| = |1 - \delta||x| \leq \lambda.
\]

Thus for every \( x \in K \)

\[
|g(x) - g(\delta x)| \leq \varepsilon/m(K).
\]

Thus \( \|g - g_\delta\| = \int_K |g - g_\delta| \leq \varepsilon \). Hence

\[
\|f - f_\delta\| \leq \|f - g\| + \|g - g_\delta\| + \|f_\delta - g_\delta\|
= (\delta^{-d} + 1)\|f - g\| + \|g - g_\delta\|
\leq (\delta^{-d} + 1)\varepsilon + \varepsilon = \varepsilon(\delta^{-d} + 2)
\]

and given \( \delta \) even closer to 1 we can make sure that \( \delta^{-d} < 2 \). So given \( \varepsilon > 0 \), we can pick \( \eta > 0 \) such that \( |1 - \delta| < \eta \) implies

\[
\|f - f_\delta\| \leq 4\varepsilon.
\]

This yields

\[
\lim_{\delta \to 1^-} \|f - f_\delta\| = 0.
\]

5. Suppose \( F \) is a closed set in \( \mathbb{R} \), whose complement has finite measure, and let \( \delta(x) \) denote the distance from \( x \) to \( F \), that is,

\[
\delta(x) = d(x, F) = \inf\{|x - y| : y \in F\}.
\]

Consider

\[
I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^d} dy.
\]

(a) Prove that \( \delta \) is continuous, by showing that it satisfies the Lipschitz condition

\[
|\delta(x) - \delta(y)| \leq |x - y|.
\]

(b) Show that \( I(x) = \infty \) for each \( x \notin F \).

(c) Show that \( I(x) < \infty \) for a.e. \( x \in F \). This may be surprising in view of the fact that the Lipschitz condition cancels only one power of \( |x - y| \) in the integrand of \( I \).

[Hint: For the last part, investigate \( f_F I(x) dx \).]

(a) For every \( z \in F \) we have

\[
|x - z| \leq |x - y| + |y - z|.
\]
Hence

\[ \delta(x) = \inf\{z \in F : |x - z|\} \leq \inf\{z \in F : |x - y| + |y - z|\} \]
\[ \leq |x - y| + \inf\{z \in F : |y - z|\} = |x - y| + \delta(y) \]

so \( \delta(x) - \delta(y) \leq |x - y| \). Changing the roles of \( x \) and \( y \), we get \( \delta(y) - \delta(x) \leq |y - x| = |x - y| \). Thus

\[ |\delta(x) - \delta(y)| \leq |x - y| \cdot \]

(b) Note that \( x \not\in F \) means that \( x \) has a ball around it disjoint from \( F \) since \( F \) is closed and hence \( \delta := d(x, F) = \delta(x) > 0 \). Thus \( |x - y| \leq \delta/2 \) implies

\[ |\delta(x) - \delta(y)| \leq \delta/2 \cdot \]

So \( \delta(y) \geq \delta/2 \). Thus

\[ \int_{x - \delta/2}^{x + \delta/2} \frac{\delta(y)}{|x - y|^2} dy \geq \int_{x - \delta/2}^{x + \delta/2} \frac{\delta/2}{|x - y|^2} dy = \int_{-\delta/2}^{\delta/2} \frac{\delta/2}{u^2} du = \delta \int_{0}^{\delta/2} \frac{du}{u^2} = \delta \cdot \infty = \infty. \]

(c) Consider the function

\[ f : F \times \mathbb{R} \to \mathbb{R} \]
\[ (x, y) \mapsto \frac{\delta(y)}{|x - y|^2}. \]

Note that \( f(x, x) = \infty \) for every \( x \in F \). \( f \) is continuous since \( \delta \) is continuous. Also clearly \( f \) is non-negative. Hence by Tonelli’s theorem, we have

\[ \int_{F} I(x) dx = \int_{\mathbb{R}} \int_{F} \frac{\delta(y)}{|x - y|^2} dxdy \]
\[ = \int_{\mathbb{R}} \delta(y) \left( \int_{F} \frac{1}{|x - y|^2} dx \right) dy \]
\[ = \int_{F^c} \delta(y) \left( \int_{F} \frac{1}{|x - y|^2} dx \right) dy. \]

Observe that for fixed \( y \in F^c \), for every \( x \in F \) we have \( \delta(y) \leq |x - y| \) and hence

\[ F \subseteq \{x \in \mathbb{R} : |x - y| \geq \delta(y)\} \]
\[ = \{x \in \mathbb{R} : x - y \geq \delta(y)\} \cup \{x \in \mathbb{R} : x - y \leq -\delta(y)\} \]
\[ = [\delta(y) + y, \infty) \cup (-\infty, y - \delta(y)] \]

and so

\[ \int_{F} \frac{1}{|x - y|^2} dx \leq \int_{y + \delta(y)}^{\infty} \frac{1}{|x - y|^2} dy + \int_{-\infty}^{y - \delta(y)} \frac{1}{|x - y|^2} dy \]
\[ = \int_{y + \delta(y)}^{\infty} \frac{1}{(x - y)^2} dy + \int_{-\infty}^{y - \delta(y)} \frac{1}{(x - y)^2} dy \]
\[ = \int_{\delta(y)}^{\infty} \frac{1}{u^2} du + \int_{-\infty}^{-\delta(y)} \frac{1}{u^2} du \]
\[ = 2 \int_{\delta(y)}^{\infty} \frac{1}{u^2} du = 2 \lim_{L \to \infty} \frac{-1}{L} \left|_{\delta(y)}^{L} \right| \]
\[ = 2 \lim_{L \to \infty} \left( \frac{1}{\delta(y)} - \frac{1}{L} \right) = \frac{2}{\delta(y)}. \]
Thus
\[ \int_F I(x)dx \leq \int_{F^c} \frac{2}{\delta(y)}dy = 2m(F^c) < \infty \]
since \( m(F^c) < \infty \) by assumption. So we showed that \( \int_F I(x)dx \) is finite hence \( I(x) \) is finite almost everywhere for \( x \in F \).

7. Let \( \Gamma \subseteq \mathbb{R}^d \times \mathbb{R}, \Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\} \), and assume \( f \) is measurable on \( \mathbb{R}^d \). Show that \( \Gamma \) is a measurable subset of \( \mathbb{R}^{d+1} \), and \( m(\Gamma) = 0 \).

Consider the function
\[
F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\
(x, y) \mapsto f(x).
\]
Given \( \alpha \in \mathbb{R} \), the set
\[ F^{-1}((\alpha, \infty)) = f^{-1}((\alpha, \infty)) \times \mathbb{R} \]
is measurable in \( \mathbb{R}^{d+1} \); hence \( G \) is a measurable function. Also the projection
\[
\mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\
(x, y) \mapsto y
\]
is measurable (in fact continuous). Thus their difference
\[
G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\
(x, y) \mapsto f(x) - y
\]
is a measurable function, thus \( G^{-1}\{0\} = \Gamma \) is a measurable subset of \( \mathbb{R}^{d+1} \). And by Corollary 2.3.3,
\[ m(\Gamma) = \int_{\mathbb{R}^d} m(\Gamma_x)dx = \int_{\mathbb{R}^d} m(\{f(x)\})dx = \int 0dx = 0. \]

10. Suppose \( f \geq 0 \), and let \( E_{2k} = \{x : f(x) > 2^k\} \) and \( F_k = \{x : 2^k < f(x) \leq 2^{k+1}\} \). If \( f \) is finite almost everywhere, then
\[ \bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\}, \]
and the sets \( F_k \) are disjoint.

Prove that \( f \) is integrable if and only if
\[ \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \quad \text{if and only if} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_{2k}) < \infty. \]

Use this result to verify the following assertions. Let
\[
f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1, \\ 0 & \text{otherwise}. \end{cases}
\]
Then \( f \) is integrable on \( \mathbb{R}^d \) if and only if \( a < d \); also \( g \) is integrable on \( \mathbb{R}^d \) if and only if \( b > d \).
Note that $F_k$’s are disjoint so letting $E = \{x : f(x) > 0\}$, as $f = 0$ outside $E$ by non-negativity we have
\[
\int f = \int_E f = \sum_{k=-\infty}^{\infty} \int_{F_k} f
\]
and by the definition of $F_k$, we have
\[
\sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) = 2 \cdot \sum_{k=-\infty}^{\infty} 2^k m(F_k)
\]
so $f$ is finite if and only if $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ is finite.

Observe that for every $k$, we have $E_{2^k} = E_{2^{k+1}} \cup F_k$

hence
\[
\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{\infty} 2^k m(E_{2^{k+1}}) + \sum_{k=-\infty}^{\infty} 2^k m(F_k)
\]
\[
= \frac{1}{2} \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_{2^{k+1}}) + \sum_{k=-\infty}^{\infty} 2^k m(F_k).
\]

Now since $\sum_{k=-\infty}^{\infty} 2^{k+1} m(E_{2^{k+1}}) = \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$, we obtain
\[
\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \frac{1}{2} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})
\]
so we get the second if and only if.

Consider the given $f$ and $E_{2^k}$’s defined by it. If $a < 0$ then $f$ is clearly integrable. So we may assume $a > 0$. Then we have
\[
E_{2^k} = \{x \in \mathbb{R}^d : |x|^{-a} > 2^k, |x| \leq 1\}
\]
\[
= \{x \in \mathbb{R}^d : |x|^a < 2^{-k}, |x| \leq 1\}
\]
\[
= \{x \in \mathbb{R}^d : |x| < \min\{1, 2^{-k/a}\}\}
\]
and since $2^{-k/a} \leq 1$ if and only if $-k/a \leq 0$ if and only if $k \geq 0$, we get
\[
E_{2^k} = \begin{cases} 
B_{\mathbb{R}^d}(1) & \text{if } k < 0 \\
B_{\mathbb{R}^d}(2^{-k/a}) & \text{if } k \geq 0
\end{cases}
\]

where $B_{\mathbb{R}^d}(r)$ denotes the ball centered at the origin of radius $r$ in $\mathbb{R}^d$. Thus
\[
\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{-1} 2^k m(B_{\mathbb{R}^d}(1)) + \sum_{k=0}^{\infty} 2^k m(B_{\mathbb{R}^d}(2^{-k/a}))
\]

Observe that a ball of radius $r$ in $\mathbb{R}^d$ contains a cube of side-length $r$ and be contained in a cube of side-length $2r$ in $\mathbb{R}^d$. Thus
\[
r^d \leq m(B_{\mathbb{R}^d}(r)) \leq (2r)^d = 2^d r^d
\]
therefore
\[ \sum_{k=-\infty}^{-1} 2^k + \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d \leq \sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq 2d \left( \sum_{k=-\infty}^{-1} 2^k + \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d \right) \]
so as \( \sum_{k=-\infty}^{-1} 2^k = 1 \), it is enough to determine when \( \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d \) converges for the integrability of \( f \). And
\[ \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d = \sum_{k=0}^{\infty} 2^{k-d/a} = \sum_{k=0}^{\infty} 2^{(1-d/a)k} \]
converges iff \( 2^{1-d/a} < 1 \) iff \( 1 - d/a < 0 \) iff \( d/a > 1 \) iff \( a < d \).
Now consider the given \( g \) and \( E_{2k} \)’s defined by it. We may define \( g \) to be 1 when \( |x| \leq 1 \) which does not affect the integrability of \( g \). If \( b < 0 \) then \( g \) is clearly not integrable. So we may assume \( b > 0 \). In this case we have
\[ E_{2k} = \{ x \in \mathbb{R}^d : |x| > 2^k, |x| > 1 \} \cup \{ x \in \mathbb{R}^d : 1 > 2^k, |x| \leq 1 \} \]
\[ = \{ x \in \mathbb{R}^d : |x|^b < 2^{-k}, |x| > 1 \} \cup \{ x \in \mathbb{R}^d : 1 > 2^k, |x| \leq 1 \} \]
\[ = \{ x \in \mathbb{R}^d : 1 < |x| < 2^{-k/b} \} \cup \{ x \in \mathbb{R}^d : 1 > 2^k, |x| \leq 1 \} . \]
Note that since \( 2^{-k/b} > 1 \) iff \( -k/b > 0 \) if \( k < 0 \); so \( E_{2k} = \emptyset \) if \( k \geq 0 \). And for \( k < 0 \) we have
\[ E_{2k} = \{ x \in \mathbb{R}^d : 1 < |x| < 2^{-k/b} \} \cup \{ x \in \mathbb{R} : |x| \leq 1 \} \]
\[ = \{ x \in \mathbb{R}^d : |x| < 2^{-k/b} \} . \]
Because \( E_{2k} = B_{\mathbb{R}^d}(2^{-k/b}) \), we have
\[ \sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d \leq \sum_{k=-\infty}^{-1} 2^k m(E_k) \leq 2d \sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d . \]
Since \( \sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d = \sum_{k=-\infty}^{-1} 2^{k(1-d/b)} = \sum_{k=1}^{\infty} 2^{(d/b-1)k} \) converges iff \( 2^{(d/b-1)} < 1 \) iff \( b > d \), the series
\[ \sum_{k=-\infty}^{\infty} 2^k m(E_{2k}) = \sum_{k=-\infty}^{-1} 2^k m(E_{2k}) \]
also converges iff \( b > d \) as desired.

**13.** Give an example of two measurable sets \( A \) and \( B \) such that \( A + B \) is not measurable.

[Hint: In \( \mathbb{R}^2 \) take \( A = \{0\} \times [0,1] \) and \( B = \mathcal{N} \times \{0\} \).]

Let \( A, B \) be as in the hint where \( \mathcal{N} \subseteq \mathbb{R} \) is a nonmeasurable set. Then
\[ A + B = \{(0, y) + (x, 0) : y \in [0,1], x \in \mathcal{N}\} = \mathcal{N} \times [0,1] \]
By Lemma 3.5, both \( A \) and \( B \) are null sets, hence are measurable. But \( A + B \) cannot be measurable, because since \( m([0,1]) = 1 > 0 \), the measurability of \( A + B \) would imply the measurability of \( \mathcal{N} \) by Proposition 3.4.
15. Consider the function defined over \( \mathbb{R} \) by
\[
f(x) = \begin{cases} 
    x^{-1/2} & \text{if } 0 < x < 1, \\
    0 & \text{otherwise}.
\end{cases}
\]
For a fixed enumeration \( \{r_n\}_{n=1}^\infty \) of the rationals \( \mathbb{Q} \), let
\[
F(x) = \sum_{n=1}^\infty 2^{-n} f(x - r_n).
\]
Prove that \( F \) is integrable, hence the series defining \( F \) converges for almost every \( x \in \mathbb{R} \). However, observe that the series is unbounded on every interval, and in fact, any function \( \tilde{F} \) that agrees with \( F \) a.e. is unbounded in any interval.

Note that
\[
\int f\chi_{(1/n,1)} = \int_1^{1/n} x^{-1/2} \, dx = 2\sqrt{x}\bigg|_{1/n}^1 = 2 - 2\sqrt{1/n}
\]
so by monotone convergence theorem
\[
\int f = \int f\chi_{(0,1)} = \lim_{n \to \infty} \left( 2 - 2\sqrt{1/n} \right) = 2.
\]
Note that by the translation invariance of the Lebesgue integral, we have
\[
\int f(x) \, dx = \int f(x-r) \, dx
\]
for any \( r \in \mathbb{R} \). Thus
\[
\int \sum_{n=1}^N 2^{-n} f(x-r_n) \, dx = \sum_{n=1}^N 2^{-n} \int f(x) \, dx = \sum_{n=1}^N 2^{-n+1} = \sum_{n=0}^{N-1} 2^{-n}
\]
and again by the monotone convergence theorem
\[
\int F = \sum_{n=0}^\infty 2^{-n} = 2.
\]
Let \( I \) be a interval. Let \( r_N \) be a rational number in \( I \). Then for every \( M > 1 \), whenever \( x \in (r_N, r_N + 2^{-2N}/M^2) \) we have \( 0 < x - r_N < 2^{-2N}/M^2 < 1 \) so since \( f \) is decreasing
\[
f(x - r_N) > f\left(2^{-2N}/M^2\right) = 2^N M.
\]
But \( (r_N, r_N + 2^{-2N}/M^2) \) intersects \( I \) in a nonempty interval which has positive measure.
So on a set of positive measure contained in \( I \), we have
\[
F(x) = \sum_{n=1}^\infty 2^{-n} f(x-r_n) \geq 2^{-N} f(x-r_N) > M.
\]
So if \( \tilde{F} \) is a.e. equal to \( F \) then \( \tilde{F} \) also has to be larger than \( M \) in a set of positive measure zero contained in \( I \). Since \( M \) was arbitrary, we get the desired conclusion.

18. Let \( f \) be a measurable finite-valued function on \([0, 1]\), and suppose that \( |f(x) - f(y)| \) is integrable on \([0, 1] \times [0, 1]\). Show that \( f(x) \) is integrable on \([0, 1]\).

We are assuming that the function
\[
F : [0, 1] \times [0, 1] \to \mathbb{R} \\
(x, y) \mapsto |f(x) - f(y)|
\]
is integrable. So by a part of Fubini’s theorem, for almost every \( y \in [0, 1] \) the function

\[
[0, 1] \to \mathbb{R} \\
x \mapsto |f(x) - f(y)|
\]
is in \( L^1([0, 1]) \). So pick and fix such a \( y \). Then the function

\[
g : [0, 1] \to \mathbb{R} \\
x \mapsto f(x) - f(y)
\]
is in \( L^1([0, 1]) \), because \(|g| \) is. Now since \( f(y) \) is finite, the constant function

\[
c : [0, 1] \to \mathbb{R} \\
x \mapsto f(y)
\]
is also in \( L^1([0, 1]) \) (its integral is \( f(y) \)). Thus \( g + c = f \in L^1([0, 1]) \).

19. Suppose \( f \) is integrable on \( \mathbb{R}^d \). For each \( \alpha > 0 \), let \( E_\alpha = \{ x : |f(x)| > \alpha \} \).
Prove that

\[
\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty m(E_\alpha)d\alpha.
\]

Consider the function

\[
F : (0, \infty) \times \mathbb{R}^d \to \mathbb{R} \\
(\alpha, x) \mapsto \begin{cases} 1 & \text{if } |f(x)| > \alpha, \\ 0 & \text{otherwise.} \end{cases}
\]

Since \( F \) is non-negative and measurable (since \( F = \chi_{|f(x)|>\alpha} \) and \( f \) is measurable). So by Tonelli’s theorem we have

\[
\int_0^\infty \int_{\mathbb{R}^d} F(\alpha, x)\,dx\,d\alpha = \int_{\mathbb{R}^d} \int_0^\infty F(\alpha, x)\,d\alpha\,dx \\
\int_0^\infty \left( \int_{\mathbb{R}^d} \chi_{E_\alpha}(x)\,dx \right) \,d\alpha = \int_{\mathbb{R}^d} \left( \int_0^\infty \chi_{(0,|f(x)|]}(\alpha)\,d\alpha \right) \,dx \\
\int_0^\infty m(E_\alpha)\,d\alpha = \int_{\mathbb{R}^d} |f(x)|\,dx.
\]