We show this by the snake lemma in the form of Lemma 1.3.2 in Weibel's book.

Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
A & \to & B & \to & \text{coker}(f) & \to & 0 \\
\downarrow g & & \downarrow f & & \downarrow \xi & & \\
0 & \to & C & \to & C & \to & 0
\end{array}
\]

The snake lemma yields an exact sequence

\[
\ker(gf) \to \ker(g) \to \ker(\xi) \to \text{coker}(gf) \to \text{coker}(g) \to \text{coker}(\xi)
\]

where \( \tilde{f} \) is induced by \( f \), i.e., is the restriction of \( f \) to \( \ker(gf) \).

Observe that

- \( \ker(\xi) = \text{coker}(f) \)
- \( \text{coker}(\xi) = 0 \)
- \( \ker(\tilde{f}) = \ker(gf) \cap \ker(f) = \ker(f) \) since \( \ker(f) \subseteq \ker(gf) \).

Therefore, inserting \( 0 \to \ker(f) \to \ker(gf) \) to the beginning of \((\dagger)\) yields the desired sequence.

I don't think you're able to assume the snake lemma here. You're being asked to prove a special case of it here.

T. Lawson said: it is OK do \( 1/4 \) use snake lemma.
2. **Möbius Strip.**

So $C_*(M)$ is of the form

$$0 \rightarrow C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \rightarrow 0$$

where $C_0(M) \cong \mathbb{Z}^4$ is the free abelian group generated by $v, w, a, b, c, d$.

And with these ordered bases, the boundary maps $\partial_1$ and $\partial_2$ are given by the matrices $B_1$,

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and $B_2$, respectively.

It is easily seen from these that the Smith Normal Form of $B_1$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\text{SNF of } B_2$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

i) $\partial_2$ is injective, so $\tilde{H}_1(M) = \ker \partial_2 = \{0\}$.

ii) $H_0(M) = \text{coker} \partial_1 \cong \mathbb{Z}$ (because in $\partial_1$, $\partial: C_0(M) \cong \mathbb{Z}^4$ to itself with $\text{rank} \partial = 1$).

iii) $\ker \partial_1$ and in $\partial_2$ are both direct sums of $C_1(M) \cong \mathbb{Z}^4$ with ranks 3 and 2, respectively. Thus $H_1(M) = \ker \partial_1/\ker \partial_2 \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}$. 

$$\cong \mathbb{Z}/2^2 \cong \mathbb{Z}$$


2. Let $X = [0,1] \times [0,1]$. The equivalence relation 
\[(0,a) \sim (1, 1-a)\]
on $X$ gives the quotient map 
$q : X \to X/_{\sim} = \mathbb{M}$

Consider the subspace $E = \{(x, 1/2) : x \in [0,1]\}$ of $X$. The map 
\[H : X \times [0,1] \to X\]
\[(x, y, t) \mapsto (x, (1-t)y + t/2)\]
defines a strong deformation retraction of $X$ to $E$.

Now $q \times \text{id} : X \times [0,1] \to \mathbb{M} \times [0,1]$ is also a quotient map since $[0,1]$ is locally compact Hausdorff.

We want to complete the diagram

\[
\begin{array}{ccc}
X \times [0,1] & \xrightarrow{H} & X \\
\downarrow q \times \text{id} & & \downarrow q \\
\mathbb{M} \times [0,1] & \to & \mathbb{M}
\end{array}
\]
into a commutative square. Since $q \times \text{id}$ is a quotient map, we only need to show that $q \circ H$ respects the identifications $q \times \text{id}$ makes.

Indeed, 
\[(q \circ H)(x, (1-t)a + t/2) = q((1, 1-a + t/2)) = q((1-t)a + t/2) = q((0, (1-t)a + t/2)) = \]

\[2 = \ldots = \underbrace{q((1-t)a + t/2)}_{x} = q((0, (1-t)a + t/2)} = \]
Indeed,
\((q \circ H)((1, 1-a), t) = q \left( H \left( (1, 1-a), t \right) \right)\)
\[= q \left( 1, (1-t)(1-a) + t/2 \right)\]
\[= q \left( 1, 1-a - t + ta + t/2 \right)\]
\[= q \left( 1, 1 - (1-t)a + t/2 \right)\]
\[= q \left( 1, 1 - [(1-t)a + t/2] \right)\]

\(\text{def of } \sim\)
\[\rightarrow q \left( 0, (1-t)a + t/2 \right)\]
\[= q \left( H \left( (0, a), t \right) \right)\]
\[= (q \circ H) \left( (0, a), t \right)\)

So there exists a cont. function
\[\tilde{H} : M \times [0, 1] \rightarrow M\]
which makes (K) commute when inserted.

Thus \(\tilde{H}\) defines a strong deformation retraction of \(M\) to \(q(E)\).

It is straightforward to check that \(\partial M = q(\partial E)\) by the defn. of the boundary of a manifold.

Thus the inclusion \(\partial M \hookrightarrow M\) is a homology equivalence.

Therefore the induced map \(H_n(\partial M) \rightarrow H_n(M)\) in homology is an isomorphism for every \(n\).

Continued at page 6
But the relative homology groups sit in a long exact sequence:

$$
\begin{align*}
\cdots & \to H_n(\Omega M) \to H_n(M) \to H_n(X) \to H_{n-1}(\Omega M) \\
& \to H_{n-1}(M) \to H_{n-1}(X) \to H_{n-2}(\Omega M) \to H_{n-2}(M) \\
& \to \cdots
\end{align*}
$$

The isomorphisms force $X$ and $\Omega$ to be the zero maps.

\text{by exactness}

Finally, again by exactness,

$$Q = m \circ m \circ \varepsilon = \ker \beta = H_n(M, \Omega M)$$

Relative homology groups completely vanish.

3. Consider the `const' functor $Z: \text{Top} \to \text{Ab}

\[ X \mapsto Z(X) \]

\[ f: X \to Y \mapsto \text{id}_{Z_Y} \]

We construct a natural transformation from $C_0$ to $Z$.

(Note that $C_0(X)$ is the free abelian group generated by $X$ as a set)

For every top. space $X$, define $E_X: C_0(X) \to Z(X) = \mathbb{Z}$, as given.

The question. (Note that $E_0: 0 \to \mathbb{Z}$ is the zero map, necessary.)

For naturality of $E_X$'s, we must show that for every continuous function $f: X \to Y$, the diagram

$$
\begin{array}{ccc}
C_0(X) & \xrightarrow{C_0(f)} & C_0(Y) \\
E_X \downarrow & & \downarrow E_Y \\
Z(X) & \xrightarrow{Z(f)} & Z(Y)
\end{array}
$$

commutes.
It is enough to check commutativity for the generators of $C_0(X)$, i.e., points of $X$: For $x \in X$,

$$
(\varepsilon_y \circ C_0(f))(x) = \varepsilon_y(f(x)) = 1 = \varepsilon_x(x) = \text{id}_Z(\varepsilon_x(x)) = (Z(f) \circ \varepsilon_x)(x)
$$

Finally, we check that the diagram composite in

$$
\begin{array}{ccc}
C_1(X) & \xrightarrow{\partial} & C_0(X) & \xrightarrow{\varepsilon} & Z
\end{array}
$$

is zero.

Indeed, $C_1(X)$ is the free abelian group on continuous functions from $\Delta = [0,1]$ to $X$ (i.e., paths in $X$) and for a path $\gamma: [0,1] \rightarrow X$, $\varepsilon(\gamma) = \gamma(1) - \gamma(0)$.

Thus, $\varepsilon \varepsilon(\gamma) = \varepsilon(\gamma(1) - \gamma(0)) = 1 - 1 = 0$.

Since $\varepsilon \varepsilon$ vanishes on generators of $C_1(X)$, $\varepsilon \varepsilon = 0$.

4. Note that $1 \in Z$ has a preimage under $\varepsilon \iff X \neq \emptyset$.

So for $X \neq \emptyset$, $\varepsilon: C_0(X) \rightarrow Z$ is an epimorphism.

Since $Z$ is a free abelian group, we can map 1 to any point of $X$ and get a splitting $1 \in Z \rightarrow C_0(X)$ of $\varepsilon$ ($\varepsilon \varepsilon = \text{id}_Z$).

Let $A = \varepsilon(Z)$. Since $\varepsilon$ is injective, $A \cong Z$. Furthermore, the splitting gives the decomposition

$$
C_0(X) = \ker \varepsilon \oplus A
$$

And since $\im \partial \subseteq \ker \varepsilon$ by (3),

$$
H_0(X) = \frac{C_0(X)}{\ker \varepsilon} = \frac{\ker \varepsilon \oplus A}{\ker \varepsilon} \cong \ker \frac{\varepsilon}{\im \partial} \oplus A
$$

$$
\cong \tilde{H}_0(X) \oplus Z.
$$
For \( X = \emptyset \), since the \( n \)-simplex \( \Delta^n \) is nonempty for all \( n \), there are no functions from \( \Delta^n \) to \( X \), let alone continuous functions. Hence \( C_n(X) = 0 \) for all \( n > 0 \).

As subquotients of \( C_n(X) \), \( H_n(X) = 0 \) for

So the reduced singular chain complex is

\[
\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0
\]

Thus

\[
\tilde{H}_n(X) = \begin{cases} 
\mathbb{Z} & \text{if } n = -1 \\
0 & \text{if } n > 0
\end{cases}
\]

4/4

2 cont'd.) Let \( Q = q(E) \). So the map

\[
f : M \longrightarrow Q \\
z \longmapsto \tilde{H}(z, 1)
\]

is a homotopy equivalence. Since \( \tilde{H}_1(Q) \) on the quotient maps

\[
\lambda : [0, 1] \longrightarrow Q \\
t \longmapsto q \left( (t, \frac{1}{2}) \right)
\]

\[
\exp : [0, 1] \longrightarrow S^1 \\
t \longmapsto e^{2\pi i t}
\]

make the same identifications, \( Q \cong S^1 \), and the equiv. class of \( \lambda \in C_1(Q) \)

\( b \) by definition of the boundary of a manifold,

\[
\partial M = q \left( \left[ 0, 1 \right] \times \{0\} \cup \left[ 0, 1 \right] \times \{1\} \right)
\]

\( Y \) has paths \( \gamma_1 : [0, 1] \rightarrow Y \) and \( \gamma_2 : [0, 1] \rightarrow Y \\
\gamma_1(t) \rightarrow \gamma_2(t)
\]

So \( \partial M \) has paths \( q \circ \gamma_1 \) and \( q \circ \gamma_2 \).
Since \((q \circ Y)(1) = q(1,0) = q(0,1) = (q \circ Y_2)(0)\), these paths can be composed to yield another path
\[ Y = Y_1 \cdot Y_2. \]
Moreover, \(Y(0) = (q \circ Y_1)(0) = q(0,0) = q(1,1) = (q \circ Y_2)(1) = \gamma(1)\), thus \(Y\) is a loop in \(\mathcal{M}\).

Clearly, as a function \(\gamma: [0,1] \rightarrow \mathcal{M}\) is surjective (\(\gamma\) hits every point in \(\gamma\)). So since \([0,1]\) is compact and \(\mathcal{M}\) is Hausdorff, \(\gamma\) is a quotient map. It makes the same identifications with \(\exp\), (\(\gamma\) has no self intersection) therefore \(\mathcal{M} \cong S^1\).

Therefore, as a chain in \(\mathbb{C}_1(\mathcal{M})\), \(\gamma\) is actually a cycle and its equivalence class \([\gamma]\) generates \(H_1(\mathcal{M})\).

So to calculate \(H_1(\mathcal{M}, \mathcal{M})\), we will chase \([\gamma]\) in
\[
H_1(\mathcal{M}) \xrightarrow{f_*} H_1(M) \xrightarrow{\sim} H_1(C) \tag{1}
\]
Here \([\gamma]\) is sent to \([f \circ \gamma]\) in \(H_1(C)\).

We investigate the loop for \(\gamma\) in \(C\):
We should unwind the definitions of \(f\) and \(\gamma\). For \(q(x, y) \in \mathcal{M}\),
\[
f(q(x, y)) = \hat{H}(q(x, y), 1)
= q(H((x, y), 1))
= q(x, \frac{1}{2})
\]
And for \(t \in [0,1]\), \(\gamma(t) = \begin{cases} (q \circ Y)(0t), & 0 \leq t < \frac{1}{2} \\ (q \circ Y)(0t_1), & \frac{1}{2} \leq t \leq 1 \end{cases}\)
Hence
\[ y(t) = \begin{cases} 
q(2t,0), & 0 \leq t \leq \frac{1}{2} \\
q(2t-1,1), & \frac{1}{2} \leq t \leq 1
\end{cases} \]

Hence
\[ (f \circ y)(t) = \begin{cases} 
f(q(2t,0)), & 0 \leq t \leq \frac{1}{2} \\
f(q(2t-1,1)), & \frac{1}{2} \leq t \leq 1
\end{cases} \]
\[ = \begin{cases} 
q(2t,1/2), & 0 \leq t \leq \frac{1}{2} \\
q(2t-1,1/2), & \frac{1}{2} \leq t \leq 1
\end{cases} \]
\[ = \begin{cases} 
\lambda(2t), & 0 \leq t \leq \frac{1}{2} \\
\lambda(2t-1), & \frac{1}{2} \leq t \leq 1
\end{cases} \]
\[ = (\lambda \cdot \lambda)(t) \] path composite

Path composition becomes addition in homology (see Lee p. 353), thus the map
\[ \theta: H_1(\mathbb{C}M) \to H_1(\mathbb{C}) \]

in (1) sends \([x]\) to \([\lambda \cdot x]\) = \([\lambda]\) + \([x]\) = 2\([x]\)

Thus the map is multiplication by 2, therefore its cokernel, which is isomorphic to \(H_1(M, \partial M)\), is
Thus $\mathbb{Z} / 2\mathbb{Z} \cong \text{coker } \theta = H_1(M, EM)$.

For $n > 1$, consider the exact sequence

$$
\cdots \longrightarrow H_2(M, EM) \longrightarrow H_2(M) \longrightarrow H_2(M, EM) \longrightarrow \cdots
$$

$$
\cdots \longrightarrow H_1(M, EM) \longrightarrow H_1(M) \longrightarrow H_1(M, EM) \longrightarrow \cdots
$$

$$
\cdots \longrightarrow H_0(M, EM) \longrightarrow H_0(M) \longrightarrow H_0(M, EM) \longrightarrow 0
$$

Thus $\ker \theta = 0$ and $\text{coker } \theta = \mathbb{Z} / 2\mathbb{Z}$. $\theta$ sits in the long exact sequence.

Here $\varphi$ is an isomorphism because $EM$ and $M$ are both path-connected and $\varphi$ is induced by the inclusion $EM \hookrightarrow M$.

Therefore $H_0(M, EM)$ is surrounded by zero groups before and after $M$. The exact sequence here is exact to 0.

For $n > 1$, $H_n(M, EM)$ is dropped in between 0 groups, so is 0.
Since $M$ and $\Omega M$ are both homotopy equivalent to $S^1$, $H_k(M) = H_k(\Omega M) = 0$ for $k \geq 2$.

Also note that the connecting homomorphism before $\Theta$ is the zero map because $\ker \Theta = 0$. So the long exact seq.

looks like

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H_0(M) & \longrightarrow & H_0(M,\Omega M) & \longrightarrow & H_1(M,\Omega M) & \longrightarrow & H_1(M) & \longrightarrow & H_1(M,\Omega M) & \longrightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H_1(M,\Omega M) & \longrightarrow & H_2(M,\Omega M) & \longrightarrow & H_3(M,\Omega M) & \longrightarrow & H_3(M) & \longrightarrow & H_3(M,\Omega M) & \longrightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
H_1(\Omega M) & \overset{\Theta}{\longrightarrow} & H_1(M) & \longrightarrow & H_1(M,\Omega M) & \longrightarrow & H_0(M,\Omega M) & \longrightarrow & H_0(M) & \longrightarrow & H_0(M,\Omega M) & \longrightarrow & 0 \\
\end{array}
\]

This $H_k(M,\Omega M) = 0$ for $k+1$ because they are trapped between zero maps in the exact seq.

And $H_1(M,\Omega M) = \text{coker} \Theta = \mathbb{Z}/2\mathbb{Z}$. 

\[\frac{4}{4}\]
5. Let \( F_n = \{ \sigma \in F : |\sigma| = n+1 \} \).

For \( 0 \leq i \leq n \), there are maps

\[
d_{i,n} : F_n \longrightarrow F_{n-1}
\]

\[
\delta = [v_0, \ldots, v_n] \longrightarrow [v_0, \ldots, \hat{v}_i, \ldots, v_n]
\]

removed.

Let \( C^{BM}_n := \mathbb{Z}^{F_n} \) — the set of all functions from \( F_n \) to \( \mathbb{Z} \) made into an abelian group by pointwise addition.

This is the same with taking formal infinite sums.

Now we define \( \Theta : C^{BM}_n \longrightarrow C^{BM}_{n-1} \). For \( f \in C^{BM}_n \), i.e., a function \( f : F_n \rightarrow \mathbb{Z} \), let

\[
\Theta(f) : F_{n-1} \longrightarrow \mathbb{Z}
\]

\[
T \longmapsto \sum_{i=0}^{\hat{\delta} \in d_{i,n}^{-1}(T)} \sum (-1)^i f(\delta)
\]

\( \Theta(f) \) is a well-defined function because on \( (n-1) \)-face \( T \) has a finite preimage under \( d_{i,n} \) since \( \delta \in d_{i,n}^{-1}(T) \) implies that \( \delta \) contains every vertex that \( T \) contains and \( (V,F) \) is locally finite.

\( \Theta \) being a group homomorphism is clear since addition is pointwise.

Observe that for \( 0 \leq j \leq i \leq n \), the diagram

\[
\begin{array}{ccc}
F_{n+1} & \xrightarrow{d_{j,n+1}} & F_n \\
\downarrow \quad d_{i,n+1} & & \downarrow d_{i,n} \\
F_n & \xrightarrow{d_{j,n}} & F_{n-1}
\end{array}
\]

commutes.
Therefore, for any \( g \in F_{n+1} = C_{n+1} \), we have
\[(\varnothing \circ \varnothing)(g)(\tau) \circ \varnothing(g) = (\varnothing(g))(\tau)\]

\[
= \sum_{i=0}^{n} \sum_{\delta \in d_{i,n}^n(\tau)} (-1)^i \varnothing g(\delta) \]

\[
= \sum_{i=0}^{n} \sum_{\delta \in d_{i,n}^n(\tau)} (-1)^i \sum_{j=0}^{n+1} \sum_{\lambda \in d_{i+1,n+1}^j(\delta)} (-1)^j g(\lambda)
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \sum_{\delta \in d_{i,n}^n(\tau)} \sum_{\lambda \in d_{i+1,n+1}^j(\delta)} g(\lambda)
\]

\[
= \sum_{0 \leq i < n} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda) + \sum_{0 \leq j < n+1} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda)
\]

\[
= \sum_{0 \leq i < n} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda) + \sum_{0 \leq j < n+1} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda)
\]

\[
= \sum_{0 \leq i < n+1} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda) + \sum_{0 \leq j < n+1} (-1)^{i+j} \sum_{\lambda \in (d_{i,n} \circ d_{i+1,n+1})^{-1}(\tau)} g(\lambda)
\]

\[
= 0.
\]

Thus \( \varnothing \circ \varnothing = 0 \).
Triangulate \( \mathbb{R} \) as follows.

So \( V = \mathbb{Z} \) with the usual ordering and \( F_i = \{ [k, k+1] : k \in \mathbb{Z} \} \). \( F_0 = \emptyset \) for \( n > 2 \).

\[ H_0^{BM}(\mathbb{R}) \text{ is the homology of the chain complex} \]

\[ \mathbb{Z}^F_2 \xrightarrow{0} \mathbb{Z}^F_3 \xrightarrow{0} \mathbb{Z}^F_4 \xrightarrow{0} \mathbb{Z}^F_5 \xrightarrow{0} \]

where \( f \in \mathbb{Z}^F_1 \), i.e. a function \( f : F_1 \rightarrow \mathbb{Z} \),

\[ \Theta f : F_0 \longrightarrow \mathbb{Z} \]

\[ k \longmapsto \sum_i \sum_{\sigma \in \delta_d^i, i}(k) \]

Since \( d_{0,1}(k) = \{ [k-1, k] \} \) and \( d_{1,1}(k) = \{ [k, k+1] \} \),

\[ \Theta f(k) = f([k-1, k]) - f([k, k+1]) \]  \( \ast \)

Write \( c_k = [k, k+1] \) short.

Claim: \( \Theta \) is surjective.

Proof: This amounts to checking that for any function \( g : \mathbb{Z} \rightarrow \mathbb{Z} \),

we can find an \( f \) which satisfies

\[ g(k) = f(c_{k-1}) - f(c_k) \]  \( \dagger \)

such an \( f \) can be defined inductively: Set \( f(c_0) = 0 \) and use \( \dagger \) to define \( f(c_1), f(c_1), \ldots \) inductively.

Thus \( H_0^{BM}(\mathbb{R}) = \operatorname{coker} \Theta = 0 \).
Claim: \( H_1^\text{BM} (\mathbb{R}) = \ker \partial \cong \mathbb{Z} \).

Proof: Let \( f \in \ker \partial \). By \((\star)\) we get
\[
\partial (e_k) = f(e_{k+1}) \quad \text{for all} \quad k \in \mathbb{Z}.
\]
Therefore \( f(e_n) = f(e_m) \) for all \( n, m \in \mathbb{Z} \).

Thus \( \ker \partial = \langle u \rangle \) where \( u \) is the function
\[
u: F_1 \rightarrow \mathbb{Z}
\]
\[
e \mapsto 1
\]
(Indeed if \( f \in \ker \partial \) and \( f(e_0) = N \), the above argument gives \( f = Nu \)).

\( u \) has infinite order in \( \mathbb{Z}^{F_1} \), so \( \langle u \rangle \cong \mathbb{Z} \). \( \square \)