1. For a space $X$, use the Mayer-Vietoris sequence to compute the homology groups of $X \times S^1$.

2. For a space $X$ with subspaces $A \subset B \subset X$, show that there is a short exact sequence of chain complexes
   \[ 0 \to C_*(B,A) \to C_*(X,A) \to C_*(X,B) \to 0. \]
   Explain how this relates the three associated types of relative homology groups.

3. Use the previous exercise to show that if $X$ is a space, $X = U \cup V$ where $U, V$ are open subsets, and $A \subset U \cap V$, there is a Mayer-Vietoris sequence relating $H_*(X,A)$, $H_*(U,A)$, $H_*(V,A)$, and $H_*(U \cap V,A)$.

4. Suppose $X$ has a sequence of subspaces $A_0 \subset A_1 \subset \cdots$ such that $X = \cup A_i$, and so that a subset $U \subset X$ is closed if and only if $U \cap A_i$ is closed for all $i$. (In this case, we say that $X$ has the direct limit topology determined by these subspaces.) Show that every element in $H_k(X)$ is the image of an element in $H_k(A_i)$ for some $i$, and that two elements in $H_*(A_i)$ become the same in $H_k X$ if and only if there is some $j \geq i$ such that their images in $H_k(A_j)$ coincide. In this case, we say $H_k(X)$ is the direct limit of the sequence of groups $H_k(A_i)$. (Hint: Show that a map $\Delta^n \to X$ always factors through some map $\Delta^n \to A_i$.)

5. Suppose $M$ is a manifold and $p \in M$. Compute the relative homology groups $H_*(M, M \setminus \{p\})$, and use it to show that “dimension” is a well-defined invariant of a connected manifold.