1. Let $M$ be an $n$-manifold and $U$ be an open subset of $M$. $U$ is Hausdorff because $M$ is Hausdorff. If $x, y \in U$ with $x \neq y$, there exist disjoint open sets $A$ and $B$ in $M$ such that $x \in A$, $y \in B$. Hence $A \cap U$ and $B \cap U$ are disjoint open sets in $U$ such that $x \in A \cap U$ and $y \in B \cap U$.

$U$ is second countable: Since $M$ is second countable, $M$ has a countable basis $\mathcal{B}$. Then it follows that $\mathcal{B}_U := \{ B \cap U : B \in \mathcal{B} \}$ is a basis for $U$ and it is countable.

$U$ is locally Euclidean: Let $x \in U$. As $M$ is an $n$-manifold, $x$ has a neighborhood $V$ homeomorphic to an open subset of $\mathbb{R}^n$, say via an embedding $\varphi : V \to \mathbb{R}^n$.

Now $U \cap V$ is open in $V$, hence $\varphi(U \cap V)$ is open in $\mathbb{R}^n$. As $\varphi(V)$ is open in $\mathbb{R}^n$, we get that $\varphi(U \cap V)$ is open in $\mathbb{R}^n$. Shortly, the whole $U \cap V$ is homeomorphic to an open subset of $\mathbb{R}^n$. As $U \cap V$ is a nbhd of $x$ in $U$, we are done.
2. Let $M_t = \{(x,y,z) \in \mathbb{R}^3 : xy^2 = t\}$.

We claim that for $t \neq 0$, $M_t$ is a 2-manifold.

First, as a subspace of $\mathbb{R}^3$, $M_t$ is Hausdorff and 2nd countable.

Let $f_t : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$(x,y,z) \mapsto xy^2 - t$$

$f_t$ is a smooth function and its Jacobian

$$J(f_t) = \begin{bmatrix} y^2 & 2xy & xy \end{bmatrix}$$

does never vanish on $M_t$ since $t \neq 0$.

Hence for every $(x,y,z) \in M_t$, $J(f_t)_{(x,y,z)}$ has rank 1.

Thus by implicit function theorem, $M_t$ is a 2-manifold.

Conversely, note that $M_0 = \{(x,y,0) : x \in \mathbb{R} \} \cup \{(0,y,0) : y \in \mathbb{R} \} \cup \{(0,0,t) : t \in \mathbb{R} \}$

So every point in $M_0$ except the origin has a nbhd homeomorphic to $\mathbb{R}^2$, but the origin has no such neighbourhood.
3. Let \( f_t : \mathbb{R}^2 \rightarrow \mathbb{R} \)
\[ (x,y) \mapsto x^2 + xy + ty^2 - t \]

\( f_t \) is smooth and \( J(f_t) = \begin{bmatrix} 2x + y & 2t + 2y \end{bmatrix} \)

Let \( M_t = \{ (x,y) \in \mathbb{R}^2 : x^2 + xy + ty^2 - t = 1 \} \) = \{ (x,y) \in \mathbb{R}^2 : f_t(x,y) = 0 \}

\( J(f_t) \) vanishes at some point in \( M_t \) only if
\[ 2x + y = 0 \]
\[ x + 2ty = 0 \]
\[ x^2 + xy + ty^2 = 1 \]

These give \( y = -2x = -2(-2t + y) = 4ty \) and \( y \neq 0 \), so \( t = 1/4 \).

Thus if \( t \neq 1/4 \), \( J(f_t) \) has rank 1 at every point of \( M_t \); and hence by IFT \( M_t \) is a 1-manifold.

Note that \( M_{1/4} = \{ (x,y) \in \mathbb{R}^2 : (x + y)^2 = 1 \} \) is not compact (it is unbounded), hence is not a closed manifold.

In general, \( M_t = \{ (x,y) \in \mathbb{R}^2 : (x + \frac{y}{2})^2 + (t - \frac{1}{4})y^2 = 1 \} \).

So we see that if \( t > 1/4 \), \( M_t \) is an ellipse; hence it is compact and therefore is a closed 1-manifold.

If \( t \leq 1/4 \) however, \( M_t \) is a hyperbola - not bounded.

So we get
\[ M_t \text{ is a closed manifold } \iff t > 1/4. \]
4. Let \( U \) be a regular coordinate ball in \( M \) (as defined in Lee).

Let \( H \) be the lower hemisphere of \( S^2 \), \( (H \cong \mathbb{R}^2) \). Now \( \partial M - U \) and \( S^2 - H \) are both manifolds with boundary and \( \partial (M - U) \cong \partial (S^2 - H) \cong S^1 \).

And \( S^2 \# M \) is the manifold without boundary when we glue \( M - U \) and \( S^2 - H \) along their boundaries.

First, observe that \( S^2 - H \cong \mathbb{D}^2 \). The inclusion \( S^1 \hookrightarrow \mathbb{D}^2 \) embeds \( S^1 \) as the boundary of the manifold \( \mathbb{D}^2 \). Also, let \( \varphi: S^1 \rightarrow \partial (M - U) \) be an embedding.

Also, let \( \Psi: S^1 \rightarrow M - U \) be an embedding such that \( \Psi(S^1) = \partial (M - U) \).

Thus, the push-out of the diagram

\[
\begin{array}{ccc}
S^1 & \rightarrow & \mathbb{D}^2 \\
\downarrow \Psi & & \downarrow \\
M - U & \rightarrow & \\
\end{array}
\]

gives \( S^2 \# M \).

Now since \( U \) is a regular coordinate ball, \( \Psi \) extends to an embedding \( \Psi: \mathbb{D}^2 \rightarrow M \) such that 
\[
\Psi(\mathbb{D}^2 - S^1) = U.
\]

So we have a commutative diagram.
We claim that this is the required push-out diagram. Indeed, if $\alpha : M-U \to X$ and $\beta : O^2 \to X$ are continuous functions such that

\[
\begin{array}{ccc}
S^1 & \rightarrow & O^2 \\
\downarrow & & \downarrow \\
M-U \xrightarrow{\alpha} & \rightarrow & M \\
\end{array}
\]

commutes, it follows that the functions

\[
\alpha : M-U \to X, \quad \beta \circ \Psi^{-1} : \Psi(O^2) \to X
\]

are continuous. Then $(M-U) \cup \Psi(O^2) = M$ and $\alpha$ and $\beta \circ \Psi^{-1}$ agree on $(M-U) \cap \Psi(O^2) = \Psi(S^1)$.

As $M-U$ and $\Psi(O^2)$ are closed subsets of $M$, $\alpha$ and $\beta \circ \Psi^{-1}$ give a continuous function

\[
\mu : M \to X
\]

And $\mu$ makes the diagram commute.

\[
\begin{array}{ccc}
S^1 & \rightarrow & O^2 \\
\downarrow & & \downarrow \\
M-U \xrightarrow{\alpha} & \rightarrow & X \\
\end{array}
\]

Clearly, $\mu$ is the unique such map. Thus $M \cong S^2 \# M$.
5. First, we observe that if we remove a disk from $P$, we get a Möbius strip:

![Diagram of Möbius strip]

where the boundary of the disk is the unmarked edges. So identifying two Möbius strips along this boundary amounts to the edge identifications:

![Diagram of edge identifications]

And this is the Klein bottle.