2. Consider the "canonical" projections

\[ \pi_i : [0, \infty) \times \mathbb{R}^{n-1} \rightarrow [0, \infty) \]

\[ \pi_2 : [0, \infty) \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \]

Define

\[ \Theta M = \left\{ x \in M \mid x \text{ has an open nbhd } U \text{ with a homeomorphism } \phi : U \rightarrow [0, \infty) \times \mathbb{R}^{n-1} \text{ such that } \pi_i \circ \phi(x) = 0 \right\} \]

We claim that \( \Theta M \) is a closed subset of \( M \). Let \( x \in \overline{\Theta M} \). Let \( U \) be an open nbhd of \( x \) in \( M \) (a differentiable manifold with boundary), there are two possibilities:

(i) \( \phi \) has an open nbhd \( V \) homeomorphic to \( \mathbb{R}^n \).

and

\[ i(M) = \left\{ x \in M \mid x \text{ has an open nbhd homeomorphic to } \mathbb{R}^n \right\} \]

\( i(M) \) is open in \( M \) because if \( x \in i(M) \) and \( U \) is an open nbhd of \( x \) homeomorphic to \( \mathbb{R}^n \) then \( U \) is an open nbhd of every point in itself we get \( U \subseteq i(M) \).

Secondly, we show that \( M = \Theta M \cup i(M) \), equivalently \( M - i(M) \subseteq \Theta M \). Let \( x \in M - i(M) \). By the assumption on \( M \), \( x \) has an open nbhd \( U \) homeomorphic, say via \( \phi \), to \( [0, \infty) \times \mathbb{R}^{n-1} \). We want to show that \( \pi_i \circ \phi(x) = 0 \). Suppose not.

Then \( \phi(x) \in (\delta, \infty) \times \mathbb{R}^{n-1} \) for some \( \delta > 0 \).

Thus \( V = \phi^{-1}(\delta, \infty) \times \mathbb{R}^{n-1} \) is an open nbhd of \( x \). But \( V \) is homeomorphic to \( (\delta, \infty) \times \mathbb{R}^{n-1} \) (via \( \phi \)), which is in turn homeo...
to $\mathbb{R}^n$. So $x \in i(M)$, which is a contradiction.

Thirdly, we show that $\Theta M \cap i(M) = \emptyset$. Suppose to the contrary that $x \in \Theta M \cap i(M)$. So there are homeomorphisms

$$\varphi : U \to \mathbb{R}^n$$

$$\psi : V \to [0, \infty) \times \mathbb{R}^{n-1}$$

where $U, V$ are open nbhds of $x$ in $M$ and $\prod_1 \psi(x) = 0$. $\psi(U \cup V)$ is an open nbhd of $\psi(x)$, so there exists $s > 0$ and an open ball $B$ in $\mathbb{R}^{n-1}$ such that

$$\psi(x) \in [0, s) \times B \subseteq \psi(U \cup V)$$

Let $W = \psi^{-1}([0, s) \times B)$. Since $[0, s) \times B \subseteq [0, \infty) \times \mathbb{R}^{n-1}$, $W$ is an open nbhd of $x$ which is homeomorphic to $[0, \infty) \times \mathbb{R}^{n-1}$ and is contained in $U$.

Therefore $\psi(W)$ is an open subset of $\mathbb{R}^n$ homeomorphic to $[0, \infty) \times \mathbb{R}^{n-1}$: a contradiction.

Thus, $\Theta M = M - i(M)$ is a closed subset of $M$. Note that $M$ is compact, so is $\Theta M$.

To show that $\Theta M$ is an $(n-1)$-dimensional manifold, we show that every point in $\Theta M$ has an nbhd homeomorphic to $\mathbb{R}^{n-1}$. Let $x \in \Theta M$. So there is a homeomorphism

$$\varphi : U \to [0, \infty) \times \mathbb{R}^{n-1}$$

where $U$ is an open nbhd of $x$ in $M$. Then $U \cup \Theta M$ is an open nbhd of $x$ in $\Theta M$, and we claim that

$$\prod_1 (\varphi(U \cup \Theta M)) = \{0\}^3.$$ 

Indeed, otherwise there would be a point $y \in U \cup \Theta M$ such that
Then arguing as above, $\Psi$ would restrict to yield a homeomorphism between an open nbhd of $\gamma$ and $\mathbb{R}^n$—a contradiction.

Thus $\Psi(U \cap \mathbb{R}M) = 0 \times 1^{\mathbb{R}^{n-1}}$. Conversely, for $q \in 1^{\mathbb{R}^{n-1}}$, since $\Psi$ is surjective, $\exists \gamma \in U$ s.t. $\Psi(\gamma) = (0, q)$. So by the definition of $\mathbb{R}M$, we get $\gamma \in \mathbb{R}M$. Hence $(0, q) \in \Psi(U \cap \mathbb{R}M)$.

$U \cap \mathbb{R}M$ is an open nbhd of $x$ in $\mathbb{R}M$ homeomorphic to $1^{\mathbb{R}^{n-1}}$.

$\mathbb{R}M$ inherits 2nd countable and Hausdorff from $M$. Why is $\mathbb{R}(\mathbb{R}M) = \mathbb{R}^3$?  

4. We need to show that if $\alpha \sim \alpha'$ (i.e., $\alpha$ and $\alpha'$ are path homotopic), and $\beta \sim \beta'$, then $\alpha \ast \beta \sim \alpha' \ast \beta'$.

Let $H : I \times I \to X$ be a path homotopy between $\alpha, \alpha'$ and $G : I \times I \to X$ be a path homotopy between $\beta, \beta'$.

By the universal property of the product topology, there exists a (unique) continuous function $H \times G$ making the diagram commute.

Let $Q = m \circ (H \times G)$. $Q$ is a continuous function from $I \times I$ to $X$ and

- $Q(s, 0) = (m \circ (H \times G))(s, 0) = m(H(s, 0), G(s, 0)) = m(\alpha(s), \beta(s)) = (\alpha \ast \beta)(s)$ for all $s \in I$,
- $Q(s, 1) = m(H(s, 1), G(s, 1)) = m(\alpha'(s), \beta'(s)) = (\alpha' \ast \beta')(s)$ for all $s \in I$. 

$$
\begin{tikzpicture}
  \node (X) at (0, 0) {$X$};
  \node (I) at (-3, 0) {$I \times I$};
  \node (XxX) at (-5.5, -2) {$X \times X$};

  \draw[->] (I) to node [above] {$H \times G$} (X);
  \draw[->] (I) to node [left] {$H$} (XxX);
  \draw[->] (I) to node [right] {$G$} (XxX);
  \draw[->] (XxX) to node [below] {$P_1$} (X);
  \draw[->] (XxX) to node [below] {$P_2$} (X);
\end{tikzpicture}
$$
\[ Q(0, t) = m(\alpha(0, t), \beta(0, t)) = m(\alpha(0), \beta(0)) \]
\[ = (\alpha \times \beta) (0) = (\alpha' \times \beta') (0) \text{ for all } (E) \]

\[ Q(1, t) = m(\alpha(1, t), \beta(1, t)) = m(\alpha(1), \beta(1)) \]
\[ = (\alpha \times \beta) (1) = (\alpha' \times \beta') (1) \text{ for all } (E) \]

Thus \( Q \) is a path-homotopy between \( \alpha \times \beta \) and \( \alpha' \times \beta' \).

5. Assume LHS is defined. Then \( \alpha(1) = \beta(0) \) and \( \gamma(1) = \delta(0) \). Then
\[ (\alpha \times \gamma)(1) = m(\alpha(1), \gamma(1)) = m(\beta(0), \delta(0)) = (\beta \times \delta)(0) \]
so RHS is also defined. Now we show their equality.

For every \( s \in I \),
\[ ((\alpha \times \beta) \times (\delta \times \delta))(s) = m((\alpha \times \beta)(s), (\delta \times \delta)(s)) \]

So for \( 0 \leq s \leq 1/2 \),
\[ ((\alpha \times \beta) \times (\delta \times \delta))(s) = m(\alpha(2s), \delta(2s)) \]
\[ = (\alpha \times \delta)(2s) \]

and for \( 1/2 \leq s \leq 1 \),
\[ ((\alpha \times \beta) \times (\delta \times \delta))(s) = m(\beta(2s-1), \delta(2s-1)) \]
\[ = (\beta \times \delta)(2s-1) \]

Thus for \( s \in I \),
\[ ((\alpha \times \beta) \times (\delta \times \delta))(s) = \begin{cases} 
(\alpha \times \delta)(2s), & 0 \leq s \leq 1/2 \\
(\beta \times \delta)(2s-1), & 1/2 \leq s \leq 1 
\end{cases} \]
3. **Main Proposition:** \( \mathbb{R} \times [0, \infty) \) is homeomorphic to \( [0, \infty) \times \mathbb{R} \).

Once we prove the main proposition (MP), it follows that for any \( p > 0 \), \( [0, \infty)^p \cong [0, \infty) \times \mathbb{R}^{p-1} \).

Hence \( [0, \infty)^p \times \mathbb{R}^{n-p} \cong [0, \infty) \times \mathbb{R}^{n-1} \times \mathbb{R}^{p-1} \cong [0, \infty) \times \mathbb{R}^{n-1} \), showing that the definitions of an "\( n \)-dimensional manifold with boundary" and "\( n \)-dimensional manifold with corners" are equivalent.

To prove MP, we identify \( \mathbb{R} \times [0, \infty) \) and consider the entire function \( f : C \to C \):

\[
 f(z) = z^2
\]

**Lemma:** \( f \) is a proper map. (preimages of compact sets are compact)

**Proof:** For any \( R > 0 \),

\[
 f^{-1} \left( B(0; R) \right) = \left\{ z \in C : \left| z^2 \right| < R \right\}
\]

\[
 = \left\{ z \in C : \left| z \right|^2 < R \right\}
\]

\[
 = \left\{ z \in C : \left| z \right| < \sqrt{R} \right\}
\]

\[
 = B(0, \sqrt{R})
\]

Therefore, \( f \) is a proper map. As \( f \) is continuous, preimages of closed sets are closed. By Heine-Borel, we are done.

Let \( A = [0, \infty) \times [0, \infty) \) and \( B = \mathbb{R} \times [0, \infty) \).

**Lemma:** \( f(A) = B \).

**Proof:** Let \( z = x + iy \in A \). Then \( f(z) = z^2 = x^2 - y^2 + i \cdot 2xy \).

Since \( x, y \in [0, \infty) \), we have \( 2xy \in [0, \infty) \) so \( f(z) \in B \).

This shows \( f(A) \subseteq B \).
Let \( w \in B \). Write \( w = re^{i\theta} \) for some \( r > 0 \), \( 0 \leq \theta \leq 2\pi \).

If \( r = 0 \), then \( w = 0 = f(0) \) so \( w \in f(A) \). If \( r \neq 0 \), since
\[
\text{Im}(re^{i\theta}) > 0,
\]
we get \( 0 < \theta < \pi \).

Thus \( z = -\sqrt{r} \cdot e^{-i\theta/2} \in A \) and \( f(z) = w \), so \( w \in f(A) \).

By the MP:

Let \( g = f|_A : A \rightarrow f(A) = B \). Again, pre-images of bounded sets under \( g \) are bounded, therefore, \( g \) is proper.

If \( g(z) = g(w) \) for \( z, w \in A \), we have \( z^2 = w^2 \).

We show that \( g \) is injective. Assume \( g(z) = g(w) \) for \( z, w \in A \).

So \( z^2 = w^2 \). If \( z = 0 \), \( w = 0 = z \). If \( z \neq 0 \), \( -z \in A \) and \( -w = z \).

Thus, \( g \) is a proper continuous, bijective function. Furthermore, by Corollary 4.97(c) in Lee's book, \( g \) is a homeomorphism.

(I probably mangled a fly here)

1. acca\(^{-1}\)bdbe\(^{-1}\)d\(^{-1}\)e\(^{-1}\)

Reorder: \( e^{-1}d^{-1}e^{-1}acca^{-1}bdbe_{acca^{-1}bdbe} \)

\( Vb \) \( \sim \) \( VW^{-1}bb : e^{-1}d^{-1}e^{-1}acca^{-1}d^{-1}bb \)

Relabel: \( edeaccac^{-1}d bb \)

Reorder: \( accac^{-1}d bbede \)

\( Vw \) \( \sim \) \( VW^{-1}ee : acca^{-1}d bbe^{-1} \ee \)

Reorder: \( cca \) acca\(^{-1}\)d bbe\(^{-1}\)d\(^{-1}\)e
Cut & paste along $b$ to get

$$aa f e^{-1} e f d d \sim a a f f d d$$

Hence the surface is

$$RP^2 \# RP^2 \# RP^2.$$ You're supposed to use orientability/ Euler characteristic to deduce this.
Cut & paste along c:

Cut & paste along f, we get

\[ ggaa'ddabbd'ee \sim ggddbbd'ee \]

Relabel: \( aabecbd'dd \)

Cut & paste along c...? Again, use orientability & Euler char.