(Note that these exercises are not identical with the corresponding ones in Lee’s text.)

1. Show that two smooth atlases for a manifold \( M \) determine the same maximal atlas if and only if their union is a smooth atlas.

2. Let \( M \) be a nonempty topological manifold of dimension \( n \geq 1 \). If \( M \) has a smooth structure, show that it has uncountably many distinct ones. (Hint: Begin by constructing homeomorphisms from the open unit disc \( \mathbb{D}^n \) to itself that are smooth on \( \mathbb{D}^n \setminus \{0\} \).)

3. Let \( N = (0, \ldots, 0, 1) \) be the “north pole” of \( S^n \subset \mathbb{R}^{n+1} \), and let \( S = -N \) be the “south pole”. Define stereographic projection \( \sigma : S^n \setminus \{N\} \to \mathbb{R}^n \) by

\[
\sigma(x^1, \ldots, x^{n+1}) = \left(\frac{x^1, \ldots, x^n}{1 - x^{n+1}}\right).
\]

Let \( \tilde{\sigma}(x) = -\sigma(-x) \) for \( x \in S^n \setminus \{S\} \).

(a) For any \( x \in S^n \setminus \{N\} \), show that \( \sigma(x) \) is the point where the line through \( N \) and \( x \) intersects the plane where \( x^{n+1} = 0 \).

(b) Show that \( \sigma \) is bijective, and

\[
\sigma^{-1}(u^1, \ldots, u^n) = \left(\frac{2u^1, \ldots, 2u^n, |u|^2 - 1}{|u|^2 + 1}\right).
\]

(c) Compute the transition map \( \tilde{\sigma} \circ \sigma^{-1} \) and verify that these two charts determine a smooth atlas on \( S^n \).

4. An angle function on a subset \( U \subset S^1 \subset \mathbb{C} \) is a continuous function \( \theta : U \to \mathbb{R} \) such that \( e^{i\theta(p)} = p \) for all \( p \in U \). Show that there exists an angle function \( \theta \) on an open subset \( U \subset S^1 \) if and only if \( U \neq S^1 \). For any such angle function, show that \((U, \theta)\) is a smooth coordinate chart for \( S^1 \) with its standard smooth structure.

5. Show that pointwise multiplication turns the set \( C^\infty(M) \) of smooth real-valued functions on \( M \) into a commutative, associative algebra over \( \mathbb{R} \).