1. Let $\mathcal{A}$ and $\mathcal{B}$ be two smooth atlases for $M$. Assume $\mathcal{A} = \mathcal{B}$. Then the maximal atlas $\mathcal{A} = \mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{B}$, hence contains $\mathcal{A} \cup \mathcal{B}$. Hence the charts in $\mathcal{A} \cup \mathcal{B}$ are smoothly compatible. Moreover, the charts in $\mathcal{A} \cup \mathcal{B}$ cover $M$, as the ones in $\mathcal{A}$ already cover $M$. Thus $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas.

Conversely, assume $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Then $\mathcal{A}$ and $\mathcal{B}$ are both smooth maximal atlases which contain $\mathcal{A}$. But $\mathcal{A}$ is contained in a unique smooth maximal atlas, hence $\mathcal{A} = \mathcal{A} \cup \mathcal{B}$. Similarly $\mathcal{B} = \mathcal{A} \cup \mathcal{B}$.

Thus $\mathcal{A} = \mathcal{B}$, i.e. $\mathcal{A}$ and $\mathcal{B}$ determine the same maximal atlas.

3. a) The line passing through $N$ and $x = (x_1', \ldots, x_m')$ can be parameterized by

$$(1-t)N + t \cdot x = (1-t)(0,0,\ldots,0,1) + t(x_1', \ldots, x_m') = (tx_1', tx_2', \ldots, tx_m', 1-t + t, 1)$$

To find where this line crosses the given plane, we solve

$$1-t + tx_m' = 0$$

$$1 = t - tx_m' = t(1-x_m')$$

$$t = \frac{1}{1-x_m'}$$

$$x_m' = 1$$

Hence the line through $N$ and $x$ intersects the given plane at

$$\left(1 - \frac{1}{1-x_m'}, \frac{1}{1-x_m'}, \ldots, \frac{1}{1-x_m'}, 0\right)$$

b) Note that for any $u = (u_1', \ldots, u_m') \in \mathbb{R}^m$, since

$$(2u_1')^2 + \ldots + (2u_m')^2 + (|u_1'|^2 - 1)^2 = 4|u|^2 + 4u_1'^2 + 2u_1'^2 + 1 = |u|^2 + 2|u|^2 + 1 = (|u|')^2$$
and \( \frac{|u|^2 - 1}{|u|^2 + 1} \neq 1 \), we have \( \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \in S^\ast \setminus \{N\} \)

Thus \( \mathcal{T} : \mathbb{R}^n \rightarrow S^\ast \setminus \{N\} \)
\[
(u^1, \ldots, u^n) \rightarrow \left( \frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, 1 \right)
\]
is a well-defined function. Now,

i) \( (\delta \circ \mathcal{T}) (u^1, \ldots, u^n) = \delta \left( \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right) \)
\[= \delta \left( \frac{\frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, 1}{1 - \frac{|u|^2 - 1}{|u|^2 + 1}} \right) \]
\[= \left( \frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, 1 \right) \]
\[= (u^1, \ldots, u^n) \]

ii) \( (\mathcal{T} \circ \delta) (x^1, \ldots, x^{n+1}) = \mathcal{T} \left( \frac{(x^1, \ldots, x^n, x^{n+1})}{1 - x^{n+1}} \right) \)
\[= \mathcal{T} \left( \frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right) \]
\[= \left( \frac{2x^1}{1 - x^{n+1}}, \ldots, \frac{2x^n}{1 - x^{n+1}}, \left( \frac{x^1}{1 - x^{n+1}} \right)^2 + \cdots + \left( \frac{x^n}{1 - x^{n+1}} \right)^2 + 1 \right) \]
\[= \left( \frac{x^1}{1 - x^{n+1}} \right)^2 + \cdots + \left( \frac{x^n}{1 - x^{n+1}} \right)^2 + 1 \]
Since \((x_1, x_2, x_n) \in S^n\) and 

\[(x_1)^2 + \ldots + (x_n)^2 = 1 - (x_{n+1})^2\]

we can write:

\[
\left( \frac{2x_1}{1-x^{n+1}}, \ldots, \frac{2x_n}{1-x^{n+1}}, \frac{(x_1)^2 + \ldots + (x_n)^2 - (1-x^{n+1})^2}{(1-x^{n+1})^2} \right)
\]

\[
= \left( \frac{2x_1}{1-x^{n+1}}, \ldots, \frac{2x_n}{1-x^{n+1}}, \frac{1 - (x_{n+1})^2 - 2x_{n+1}(x_{n+1})^2}{(1-x^{n+1})^2} \right)
\]

\[
= \left( \frac{2x_1}{1-x^{n+1}}, \ldots, \frac{2x_n}{1-x^{n+1}}, \frac{2x_{n+1}}{(1-x^{n+1})^2} \right)
\]

Thus \(\tilde{\sigma}\) is bijective and \(T = \sigma^{-1}\).

c) The transition map is the composition (since \(S^n \setminus \{N,S\} \cong \mathbb{R}^n \setminus \{0\}\))

\[
\delta^{-1}: S^n \setminus \{N,S\} \rightarrow \mathbb{R}^n \setminus \{0\}
\]

\[
\delta^{-1} \left( S^n \setminus \{N,S\} \right) = \mathbb{R}^n \setminus \{\delta^{-1}(S)\}
\]

\[
\delta\left( S^n \setminus \{N,S\} \right) = \mathbb{R}^n \setminus \{\delta(N)\}
\]

\[
= \mathbb{R}^n \setminus \{0\}
\]
So the transition map is given by

\[
\begin{align*}
\mathbb{R}^n \setminus \{0\} & \rightarrow \mathbb{R}^n \setminus \{0\} \\
u = (u^1, \ldots, u^n) & \mapsto \tilde{\sigma}^{-1} \left( \frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, \frac{1}{|u|^2 + 1} \right) \\
& = \tilde{\sigma}^{-1} \left( \frac{-2u^1}{|u|^2 + 1}, \ldots, \frac{-2u^n}{|u|^2 + 1}, \frac{1}{|u|^2 + 1} \right) \\
& = \left( \frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1} \right)
\end{align*}
\]

which is clearly a smooth function. So the two charts \((S^n \setminus \{N\}, \tilde{\sigma})\) and \((S^n \setminus \{S\}, \tilde{\sigma}^2)\) are smoothly compatible. Clearly they also cover \(S^n\), hence they determine a smooth atlas on \(S^n\).

What about \((\tilde{\sigma}^{-1})^{-1}\)?
4. Consider the covering map \( \exp : \mathbb{R} \to S^1 \). Now the first part of the problem can be restated as follows: Show that the inclusion map \( U \to S^1 \) can be lifted to \( \mathbb{R} \) if and only if \( U \neq S^1 \).

\[
\theta : U \to \mathbb{R}
\]

If \( U = S^1 \), the above means that the identity map \( \text{id} : S^1 \to S^1 \) factors through \( \mathbb{R} \) and that is impossible (applying \( \text{H}_1 \), we get that \( \pi_1 S^1 \approx \mathbb{Z} \) is the zeroth group, for instance).

If \( U \neq S^1 \), we first show that \( U \) is simply connected: Pick \( \rho \in S^1 \setminus U \), so \( U \) is an open subset of \( S^1 \setminus \{ \rho \} \). But \( S^1 \setminus \{ \rho \} \) is homeomorphic with \( \mathbb{R} \), via stereographic projection and open subsets of \( \mathbb{R} \) are disjoint unions of open intervals.

So for a component \( U_0 \) of \( U \), \( U_0 \) is simply connected and (as an open subset of \( S^1 \)) \( U_0 \) is locally path connected. Thus by covering space theory (Corollary 11.19 in Lee's Intro to Top Manifolds), the inclusion of \( U_0 \) in \( S^1 \) can be lifted to \( \mathbb{R} \) along \( \exp \). Lifting each of the components like this, we get a lifting of \( U \).

For the second part, we observe the following:

- \( \theta(U) \) is open in \( \mathbb{R} \): Let \( U_0 \) be a (connected) component of \( U \). It is enough to show that \( \theta(U_0) \) is open in \( \mathbb{R} \) (because \( \theta(U) = \bigcup_{U_0 \in U} \theta(U_0) \)).

  As \( U_0 \) is connected, \( \theta(U_0) \subset \mathbb{R} \) is connected, hence \( \theta(U_0) \) is an interval.

  We showed above that via stereographic projection, \( \theta \) is a homeomorphism onto its image. Since \( e^{i \theta(p)} = p \) for all \( p \in U \), the map \( \theta(U) \to U \) is a well-defined continuous inverse of \( \theta : U \to \mathbb{R} \).

- \( \theta(U) \) is open in \( \mathbb{R} \): Let \( U_0 \) be a (connected) component of \( U \). It is enough to show that \( \theta(U_0) \) is open in \( \mathbb{R} \) (because \( \theta(U) = \bigcup_{U_0 \in U} \theta(U_0) \)).
On one hand, $\Theta(U_0)$ is homeomorphic to $U_0$ and we showed above via stereographic projection that $U_0$ is homeomorphic to an open interval in $\mathbb{R}$.

On the other hand, $\Theta(U_0)$ is a connected subset of $\mathbb{R}$, hence is an interval.

Therefore, being homeomorphic to an open interval, $\Theta(U_0)$ itself must be an open interval because other kinds of intervals in $\mathbb{R}$ contain a point which can be removed without destroying connectivity.

- The above arguments show that $(U, \theta)$ is a coordinate chart on $S^1$. Now we show that it is compatible with the smooth structure on $S^1$. By Example 1.31 in Lee's book Introduction to Smooth Manifolds, 

$$\left\{ (U_1^+, \psi_1^+), (U_2^+, \psi_2^+), (U_1^-, \psi_1^-), (U_2^-, \psi_2^-) \right\}$$

is a smooth atlas on $S^1$ where

$$U_1^+ = \left\{ (x, \gamma) \in S^1 : x > 0 \right\} \quad U_2^+ = \left\{ (x, \gamma) \in S^1 : \gamma > 0 \right\}$$

$$U_1^- = \left\{ (x, \gamma) \in S^1 : x < 0 \right\} \quad U_2^- = \left\{ (x, \gamma) \in S^1 : \gamma < 0 \right\}$$

$$\psi_1^+ : U_1^+ \longrightarrow (-1, 1) \quad \psi_2^+ : U_2^+ \longrightarrow (-1, 1)$$

$$\begin{array}{c}
(x, \gamma) \mapsto \gamma \\
(x, \gamma) \mapsto x
\end{array}$$

$$\psi_1^- : U_1^- \longrightarrow (-1, 1) \quad \psi_2^- : U_2^- \longrightarrow (-1, 1)$$

$$\begin{array}{c}
(x, \gamma) \mapsto \gamma \\
(x, \gamma) \mapsto x
\end{array}$$

Let’s check that $(U, \theta)$ and $(U_i^+, \psi_i^+)$ are smoothly compatible.

We want to show that the map $\Theta(U_1^+ \cap U) \to \psi_1^+(U_1^+ \cap U)$ is smooth with a smooth inverse.

$$\psi_1^+(U_1^+ \cap U) \xrightarrow{\psi_1^+ \circ \Theta} \psi_1^+(\Theta^{-1}(U_1^+ \cap U))$$
Note that since $\theta$ is an angle function,
\[ \theta^{-1}(c^t) = e^{i \theta(\theta^{-1}(c^t))} = e^{i c^t} \]
So
\[ \Phi_t^\circ (\theta^{-1}(c^t)) = \Phi_t^\circ (e^{i c^t}) = \Phi_t^\circ (\cos, \sin) = \sin t \]
Thus the homeomorphism $\Phi_t^\circ \circ \theta^{-1}$ is just taking sine, hence is smooth.
Moreover since $\sin (\Phi_t^\circ \circ \theta^{-1}) \leq (-1,1)$ the derivative cost is never zero, therefore $\Phi_t^\circ \circ \theta^{-1}$ has a smooth inverse.
Similarly, we get
\[ \Phi_t^\circ \circ \theta^{-1} = \sin \left( \int_0^{u_t^\circ \circ \theta^{-1}} \right) \]
\[ \Phi_t^\circ \circ \theta^{-1} = \cos \left( \int_0^{u_t^\circ \circ \theta^{-1}} \right) \]
\[ \Phi_t^\circ \circ \theta^{-1} = \cos \left( \int_0^{u_t^\circ \circ \theta^{-1}} \right) \]
which are all smooth with nonvanishing derivatives, so they

5. First we observe that for any set $X$, the set of all functions $\mathbb{R}^X$ is a commutative associative algebra with

\[
\begin{align*}
(f \cdot g)(x) &= f(x) \cdot g(x) \\
(f + g)(x) &= f(x) + g(x) \\
(\lambda f)(x) &= \lambda f(x)
\end{align*}
\]

The axioms of an $\mathbb{R}$-algebra directly follow from the fact that $\mathbb{R}$ itself is an $\mathbb{R}$-algebra. For instance, to check associativity of $\cdot$, we observe

\[
(f \cdot (g \cdot h))(x) = \left(f \cdot \left((g \cdot h)(x)\right)\right)(x) = f(x) \cdot (g(x) \cdot h(x)) = f(x) \cdot (g(x) \cdot h(x)) = f(x) \cdot (g(x) \cdot h(x))
\]

Or for distributivity, we check

\[
(f \cdot (g + h))(x) = f(x) \cdot (g(x) + h(x)) = f(x) \cdot g(x) + f(x) \cdot h(x)
\]

for all $x \in X$, so $f \cdot (g + h) = f \cdot g + f \cdot h$

and so on. This is nothing but the construction of the direct product $\mathbb{R}^X$ as an $\mathbb{R}$-algebra.
For a smooth manifold $M$, $C^\infty(M)$ is clearly a subset of $\mathbb{R}^M$. We show that it is actually a subalgebra of $\mathbb{R}^M$ by checking closure under multiplication, addition and scalar multiplication:

1. Let $f, g \in C^\infty(M)$, i.e. $f : M \to \mathbb{R}, g : M \to \mathbb{R}$ are smooth functions. Then the function
   \[
   \langle f, g \rangle : M \to \mathbb{R}^2, \quad x \mapsto (f(x), g(x))
   \]
   is also smooth (by proposition 2.12 in Lee's book). Moreover, the addition and multiplication functions
   \[
   + : \mathbb{R}^2 \to \mathbb{R}, \quad (a, b) \mapsto a + b
   \]
   \[
   \cdot : \mathbb{R}^2 \to \mathbb{R}, \quad (a, b) \mapsto ab
   \]
   are smooth functions. Therefore the compositions
   \[
   M \xrightarrow{\langle f, g \rangle} \mathbb{R}^2 \xrightarrow{+} \mathbb{R} \quad M \xrightarrow{\langle f, g \rangle} \mathbb{R}^2 \xrightarrow{\cdot} \mathbb{R}
   \]
   are also smooth. We see that the first function is nothing but $f + g$ and the second function is $fg$.

   Thus $f + g, fg \in C^\infty(M)$.

2. For scalar multiplication, note that for any $\lambda \in \mathbb{R}$ the function
   \[
   \cdot_\lambda : \mathbb{R} \to \mathbb{R}, \quad a \mapsto \lambda a
   \]
   is smooth. Therefore given $f \in C^\infty(M)$, $\lambda f$ is nothing but the composition
   \[
   M \xrightarrow{f} \mathbb{R} \xrightarrow{\cdot_\lambda} \mathbb{R} \quad \text{Thus } \lambda f \in C^\infty(M).
   \]
2. Let \( \mathcal{A} = \{(U_k, \varphi_k) : k \in I \} \) be a smooth atlas of \( M \).

(B.7 Lemma 1.10 in Lee) \( M \) has a basis of coordinate balls.

So by restricting to these coordinate balls, we may assume that \( \varphi_k \) is a homeomorphism from \( U_k \) to the unit disk \( \mathbb{D}^n \).

Moreover, since \( \mathbb{D}^n \) is diffeomorphic to \( \mathbb{R}^n \) via \( x \mapsto \frac{x}{1 + \|x\|^2} \), we may assume that \( \varphi_k \) is a homeomorphism onto \( \mathbb{R}^n \) (since composing \( \varphi_k \) with the above map retains smoothness).

Now, since \( M \) is paracompact we may assume that the open cover \( \{U_k : k \in I\} \) of \( M \) is locally finite.

To see this, apply Theorem 1.15 in Lee to \( M \) with \( X = \{U_k : k \in I\} \) and \( \mathcal{B} = \{ \text{coordinate balls in } M \} \).

Lemma: \( \mathcal{A} \) has a minimal subset which covers \( M \).

Proof: Let \( \mathcal{I} = \{ \mathcal{B} \subset \mathcal{A} : \mathcal{B} \text{ covers } M \} \) partially ordered by reverse inclusion. Observe that

- \( \mathcal{I} \neq \emptyset \), so \( \mathcal{I} \) is nonempty.

- Let \( (\mathcal{B}_k)_{k \in A} \) be a chain in \( \mathcal{I} \). We show that \( \mathcal{B} = \bigcap_{k \in A} \mathcal{B}_k \in \mathcal{I} \), i.e. \( \mathcal{B} \) covers \( M \).

Take any \( x \in M \) and define \( \mathcal{B}^x = \{(U, \varphi) \in \mathcal{B} : x \in U\} \).

We want to show \( \mathcal{B}^x \neq \emptyset \). Note that if we similarly define \( \mathcal{B}_k^x = \{(U, \varphi) \in \mathcal{B}_k : x \in U\} \), we have

\[
\mathcal{B}^x = \bigcap_{k \in A} \mathcal{B}_k^x
\]

But each \( \mathcal{B}_k^x \) is a finite set since \( \mathcal{B}_k \) is a locally finite cover of \( M \). Thus, being a decreasing sequence of nonempty finite sets (since \( \mathcal{B}_k^x \subset \mathcal{B}_k \) if \( \mathcal{B}_k \subset \mathcal{B}_k \)), \( \mathcal{B}^x \) is nonempty.

By Zorn's lemma, we are done.

\( \square \)
Let $\mathcal{B}$ be this minimal subset. Since $\mathcal{B}$ covers $M$, and the charts in $\mathcal{B}$ are smoothly compatible because they are already in $\mathcal{A}$, hence $\mathcal{B}$ is a smooth atlas for $M$.

We also make the following observation: Every $(U_\alpha, \psi_\alpha)$ in $\mathcal{B}$ contains a point which is not contained in any other chart. Because otherwise $\mathcal{B} = \{(U_\alpha, \psi_\alpha)\}$ would still cover $M$, contradicting the minimality of $\mathcal{B}$.

We did all of the above to verify that we can proceed with the following assumption:

Assumption: $\mathcal{A} = \{(U_\alpha, \psi_\alpha) : \alpha \in J\}$ is a smooth atlas of $M$ such that each $\psi_\alpha$ is a homeomorphism onto $\mathbb{R}^n$ and each $U_\alpha$ contains a point $x_\alpha$ which is not contained in any other $U_\beta$.

Now we construct homeomorphisms of $\mathbb{R}^n$ to itself to which we will use to alter $\mathcal{A}$ &

For every $s \in (0, \infty)$ consider the map

$$F_s : \mathbb{R}^n \to \mathbb{R}^n$$

$$x \mapsto \begin{cases} \frac{|x|^{s-1}}{|x|^s} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since the Euclidean norm $\|x\|_2 \to |x|$ is smooth on $\mathbb{R}^n - \{0\}$, so is $F_s$. Note that $\lim_{x \to 0} \frac{\|F_s(x)\|_2}{\|x\|_2} = \lim_{x \to 0} \frac{|x|^s}{\|x\|^s} = 0 \Rightarrow F_s(0)$, so $F_s$ is continuous everywhere.

Claim: $F_s \circ F_t = F_{st}$.

Proof: $(F_s \circ F_t)(x) = F_s(F_t(x)) = F_s(\frac{|x|^{t(s-1)}}{|x|^t} x) = \frac{|x|^{t(s-1)}}{|x|^s} \frac{|x|^{s-1}}{|x|^t} x = \frac{|x|^{ts-t(s-1)}}{|x|^t} = F_{ts-t(s-1)}(x) = F_{st}(x)$.

For $x = 0$ $(F_s \circ F_t)(0) = 0 = F_{st}(0)$.

\[\Box\]
As a result, since $F_1 = \text{id}$ for every $s \in (0, \infty)$, $F_s$ is a homeomorphism with $(F_s)^{-1} = F_{1/s}$.

Claim: For $s \in (0, 1)$, $F_s$ is not differentiable (hence not smooth) at the origin.

Proof: If $F_s$ were differentiable, the function given by the composition

$$f: \mathbb{R} \longrightarrow \mathbb{R}^n \xrightarrow{F_s} \mathbb{R}^n \xrightarrow{(x_0, \ldots, x_n) \longmapsto x_1} \mathbb{R},$$

would be differentiable at 0.

But for $x \neq 0$, $f(x) = 1x_1^{s-1} x$ and so

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1h^{s-1} h}{h} = \lim_{h \to 0} h^{s-1} = \infty \text{ since } s < 0.$$ 

Now, for each $s \in (0, 1)$, let $\mathcal{A}_s = \{(U_s, F_s \circ \psi_s) : \alpha \in I\}$. That is, we compose every chart

For every $\alpha \in I$, define $t_\alpha: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$t_\alpha \circ F_s \circ \psi_s \circ t_\alpha^{-1} \circ \psi_\alpha^{-1}$$

t_\alpha is clearly a diffeomorphism, it is just a translate.

Now for each $\beta \in \mathcal{A}_0$, let $\mathcal{A}_s = \{(U_\alpha, F_s \circ \psi_\alpha) : \alpha \in I\}$. 

Proposition: For every $s \in (0, 1)$, $\mathcal{A}_s$ is a smooth atlas on $M$.

Proof: We didn't change the $U_\alpha$'s in $\mathcal{A}_0$, so $\mathcal{A}_s$ also covers $M$.

Write $\psi_{s, \alpha} = F_s \circ t_\alpha \circ \psi_\alpha$. Since each $F_s, t_\alpha, \psi_\alpha$ is a homeomorphism onto $\mathbb{R}^n$, so is $\psi_{s, \alpha}$.

Finally, we show that the charts in $\mathcal{A}_s$ are smoothly compatible.

So we look at the diagram

$$
\begin{align*}
    & U_{s, \alpha} \\
\rightarrow & U_{s, \alpha} \cap U_{s, \beta} \\
\rightarrow & U_{s, \beta}
\end{align*}
$$

Finally, we show that the charts in $\mathcal{A}_s$ are smoothly compatible.
Spelling this out, we have

\[
\begin{align*}
& (F_5 \circ t_{\beta} \circ \psi_{\beta}) \circ (F_5 \circ t_{\alpha} \circ \psi_{\alpha})^{-1} \\
= & F_5 \circ t_{\beta} \circ \psi_{\beta} \circ (t_{\alpha} \circ \psi_{\alpha})^{-1} \circ F_5^{-1} \quad (\star)
\end{align*}
\]

Note that by our assumption, \( x_\alpha \notin U_\beta \). Thus \( \psi_{\alpha}(x_\alpha) \notin \psi_{\beta}(U_\alpha \cap U_\beta) \), and hence \( \bar{\varnothing} = t_{\alpha}(\psi_{\alpha}(x_\alpha)) \notin (t_{\alpha} \circ \psi_{\alpha})(U_\alpha \cap U_\beta) \), and hence

\[
F_5 \circ \bar{\varnothing} = F_5(t_{\alpha}(\psi_{\alpha}(x_\alpha))) \notin (F_5 \circ t_{\alpha} \circ \psi_{\alpha})(U_\alpha \cap U_\beta).
\]

And since \( x_\beta \notin U_\alpha \), we have \( \bar{\varnothing} \notin (t_{\beta} \circ \psi_{\beta})(U_\alpha \cap U_\beta) \).

Therefore, in the long composition in (\star), \( \bar{\varnothing} \) is not in the domain of \( F_5^{-1} \) and not in the domain \( F_5 \). Thus these maps are smooth on their domains, so the whole composition is smooth, as \( \psi_{\beta} \circ \psi_{\alpha}^{-1} \) is smooth since \( \alpha \) is a smooth atlas and \( t_{\alpha}, t_{\beta} \) are already smooth.

\[\square\]

Proposition: For \( s,t \in (0,1) \) \( U_s \) and \( U_t \) are not smoothly compatible atlases on \( M \) unless \( s = t \).

Proof: WLOG, say \( s > t \). For an arbitrary \( \alpha \in \mathcal{I} \), we show that the charts \( (U_\beta, \psi_{s, \alpha}) \in A_s \) and \( (U_\beta, \psi_{t, \alpha}) \in A_t \) are not compatible.
Here, the diagram to look at is

\[
\begin{array}{c}
\begin{array}{c}
\in\downarrow \quad \in\downarrow \\
U_{k, \ell} \quad U_{k, \ell} \\
\downarrow t_k \quad \downarrow t_k \\
F_\ell \quad F_\ell \\
\downarrow \quad \downarrow \\
\mathbb{R}^n \quad \mathbb{R}^n \\
\end{array}
\end{array}
\end{array}
\]

Since

\[
\begin{align*}
U_{k, \ell} \circ U_{k, s}^{-1} &= (F_\ell \circ t_k \circ U_{k, \ell}) \circ (F_\ell \circ t_k \circ U_{k, s})^{-1} \\
&= F_\ell \circ t_k \circ (U_{k, \ell}^{-1} \circ t_k^{-1} \circ F_\ell^{-1}) \\
&= F_\ell \circ F_\ell^{-1} \\
&= F_\ell \circ F_\ell^{-1} \\
&= F_\ell / s
\end{align*}
\]

and since \( t/s < 1 \), the transition map \((F_\ell / s)\) is not smooth on its domain by a previous claim.

Thus \( \{ U_s : s \in \mathbb{R} \} \) is an uncountable collection of atlases on \( M \) s.t. none of them are compatible with each other.