Math 8302, Manifolds and Topology II
Auxiliary problems
Due in-class on Monday, April 29

The interaction between these problems and the final is as stated in class.

1. Suppose $M$ is a smooth manifold of dimension $n$. A \textit{smoothly time-dependent} 1-form on $M$ is a family of 1-forms $\omega(t)$ for $t \in (a, b)$ such that, in any coordinate chart $(x^1, \ldots, x^n)$, the coordinate expression of $\omega$ is of the form
   \[ \sum_{k=1}^{n} f_k(x^1, \ldots, x^n, t) \ dx^k \]
   where each $f_k$ is a smooth function of $(x^1, \ldots, x^k, t)$.
   Give (and prove) the most concise condition that you can find for $\omega$ to satisfy the following:
   \textit{For any time interval $[c, d] \subset (a, b)$ and any two smooth curves $\gamma_1, \gamma_2 : [c, d] \to M$, the line integrals of $\omega(t)$ over $\gamma_1$ and $\gamma_2$ agree.}

   Note that such a line integral involves the change of $\gamma_i(t)$ and substituting the time $t$ into the 1-form $\omega(t)$. For example, such a line integral would be calculated in coordinates as
   \[ \sum_{k=1}^{n} \int_{c}^{d} f_k(x^1(t), \ldots, x^n(t), t) \frac{dx^k}{dt} \ dt \]

2. For $n > 0$, the manifold $\mathbb{CP}^n$ contains $\mathbb{CP}^1$ as a submanifold, and the second de Rham cohomology group $H^2_{dR}(\mathbb{CP}^n)$ is $\mathbb{R}$. Find, for each $n$, a generator of $H^2_{dR}$ in the following way: give an explicit \textit{closed} 2-form, take the integral over $\mathbb{CP}^1$, and show that this integral is nonzero.
   Do the same for the complex Grassmannian $Gr_{\mathbb{C}}(2, 4)$.

3. Track down the definition of the Moore manifold from Royden’s real analysis text. Prove that it is a manifold. (Unless you’re reading someone’s transcription, there is probably not a mistake in the problem.)